A Appendix: Detail calculation for Sections 2 and 3

This part is a supplement to Sections 2 and 3 and contains detailed proof for referees' convenience(only for review process).

A.1 Calculation for the proof of Theorem 2.2

Verification of A_1 :

$$\begin{aligned} A_1 = & \sum_{c_1, c_3, c_4=1}^{n} \sum_{c_2=1}^{p} x_{c_1c_2}\psi_{c_3} o_{jc_3} \left\{ \sum_{b \neq c_3}^{n} \frac{o_{c_2b}\{o_{ib}o_{c_4c_3}x_{c_1c_4} + o_{c_4b}o_{ic_3}x_{c_1c_4}\}}{\ell_{c_3} - \ell_b} \right. \\ & \left. + \sum_{b=n+1}^{p} \frac{o_{c_2b}\{o_{ib}o_{c_4c_3}x_{c_1c_4} + o_{c_4b}o_{ic_3}x_{c_1c_4}\}}{\ell_{c_3}} \right\} \\ = & \sum_{c_3=1}^{n} \left\{ \sum_{b \neq c_3}^{n} \frac{\psi_{c_3}}{\ell_{c_3} - \ell_b} \big(o_{ib}o_{jc_3}\{\boldsymbol{O}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\}_{bc_3} + o_{ic_3}o_{jc_3}\{\boldsymbol{O}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\}_{bb} \big) \right. \\ & \left. + \sum_{b=n+1}^{p} \frac{\psi_{c_3}}{\ell_{c_3}} \big(o_{ib}o_{jc_3}\{\boldsymbol{O}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\}_{bc_3} + o_{ic_3}o_{jc_3}\{\boldsymbol{O}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_2\}_{bb} \big) \right\} \\ = & \sum_{c_3=1}^{n} o_{ic_3}o_{jc_3} \sum_{b \neq c_3}^{n} \frac{\ell_b\psi_{c_3}}{\ell_{c_3} - \ell_b}. \end{aligned}$$

□

Verification of A_2 :

$$\begin{aligned} A_2 = & \sum_{c_1, c_3, c_4=1}^{n} \sum_{c_2=1}^{p} x_{c_1c_2}\psi_{c_3} o_{c_2c_3} \left\{ \sum_{b \neq c_3}^{n} \frac{o_{jb}\{o_{ib}o_{c_4c_3}x_{c_1c_4} + o_{c_4b}o_{ic_3}x_{c_1c_4}\}}{\ell_{c_3} - \ell_b} \right. \\ & \left. + \sum_{b=n+1}^{p} \frac{o_{jb}\{o_{ib}o_{c_4c_3}x_{c_1c_4} + o_{c_4b}o_{ic_3}x_{c_1c_4}\}}{\ell_{c_3}} \right\} \\ = & \sum_{b=1}^{n} \sum_{c_3 \neq b}^{n} \frac{\psi_{c_3}}{\ell_{c_3} - \ell_b} \big(o_{ib}o_{jb}\{\boldsymbol{O}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\}_{c_3c_3} + o_{ic_3}o_{jb}\{\boldsymbol{O}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\}_{c_3b} \big) \\ & + \sum_{b=n+1}^{p} \sum_{c_3=1}^{n} \frac{\psi_{c_3}}{\ell_{c_3}} \big(o_{ib}o_{jb}\{\boldsymbol{O}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\}_{c_3c_3} + o_{ic_3}o_{jb}\{\boldsymbol{O}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_2\}_{c_3b} \big) \\ = & \sum_{b=1}^{n} o_{ib}o_{jb} \sum_{c_3 \neq b}^{n} \frac{\ell_{c_3}\psi_{c_3}}{\ell_{c_3} - \ell_b} + \sum_{b=n+1}^{p} o_{ib}o_{jb} \sum_{c_3=1}^{n} \psi_{c_3}. \end{aligned}$$

$\square$

Verification of A_3 :

$$A_3 = \sum_{c_1,\,c_3,\,c_4,\,m=1}^{n} \sum_{c_2=1}^{p} x_{c_1c_2} o_{c_2c_3} o_{jc_3} \frac{\partial \psi_{c_3}}{\partial \ell_m} o_{c_4m} o_{im} x_{c_1c_4}$$
$$= 2 \sum_{c_3,\,m=1}^{n} o_{im} o_{jc_3} \frac{\partial \psi_{c_3}}{\partial \ell_m} \{\boldsymbol{O}_1' \boldsymbol{X}' \boldsymbol{X} \boldsymbol{O}_1\}_{c_3m} = 2 \sum_{c_3=1}^{n} o_{ic_3} o_{jc_3} \ell_{c_3} \frac{\partial \psi_{c_3}}{\partial \ell_{c_3}}.$$

$\square$

A.2 Calculation for the proof of Theorem 2.3

Verification of $T_{ic_3}^{(11)}$:

$$\begin{aligned}
T_{ic_3}^{(11)} &= \sum_{c_1,\,c_5=1}^{n} \sum_{c_2=1}^{p} x_{c_1i} \psi_{c_5} o_{c_3c_5} \Big\{ \sum_{b\neq c_5}^{n} \sum_{c_6=1}^{p} \frac{o_{c_2b}\{o_{c_2b}o_{c_6c_5}x_{c_1c_6} + o_{c_6b}o_{c_2c_5}x_{c_1c_6}\}}{\ell_{c_5} - \ell_b} \\
&\qquad + \sum_{b=n+1}^{p} \sum_{c_6=1}^{p} \frac{o_{c_2b}\{o_{c_2b}o_{c_6c_5}x_{c_1c_6} + o_{c_6b}o_{c_2c_5}x_{c_1c_6}\}}{\ell_{c_5}} \Big\} \\
&= \{\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1 \mathbf{Diag}\big(\sum_{b\neq 1} \frac{\psi_1}{\ell_1 - \ell_b}, \ldots, \sum_{b\neq n} \frac{\psi_n}{\ell_n - \ell_b}\big)\boldsymbol{O}_1'\}_{ic_3} \\
&\quad + (p-n)\{\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1 \mathbf{Diag}\big(\frac{\psi_1}{\ell_1}, \ldots, \frac{\psi_n}{\ell_n}\big)\boldsymbol{O}_1'\}_{ic_3} \\
&= \{\boldsymbol{O}_1 \mathbf{Diag}\big(\sum_{b\neq 1} \frac{\ell_1\psi_1}{\ell_1 - \ell_b}, \ldots, \sum_{b\neq n} \frac{\ell_n\psi_n}{\ell_n - \ell_b}\big)\boldsymbol{O}_1'\}_{ic_3} + (p-n)\{\boldsymbol{O}_1 \mathbf{Diag}\big(\psi_1, \ldots, \psi_n\big)\boldsymbol{O}_1'\}_{ic_3}.
\end{aligned}$$

The second equality can be seen as follows: Summing over c_2 from 1 to p, we see that $\sum_{c_2=1}^{p} o_{c_2b} o_{c_2b} = \delta_{bb}$ and $\sum_{c_2=1}^{p} o_{c_2b} o_{c_2c_5} = \delta_{bc_5} = 0$ for $c_5 \neq b$. $\square$

Verification of $T_{ic_3}^{(12)}$:

$$\begin{aligned} T_{ic_3}^{(12)} &= \sum_{c_1,\,c_5=1}^{n}\sum_{c_2=1}^{p} x_{c_1 i}\psi_{c_5} o_{c_2c_5}\Big\{\sum_{b\neq c_5}^{n}\sum_{c_6=1}^{p}\frac{o_{c_3b}\{o_{c_2b}o_{c_6c_5}x_{c_1c_6}+o_{c_6b}o_{c_2c_5}x_{c_1c_6}\}}{\ell_{c_5}-\ell_b} \\ &\qquad + \sum_{b=n+1}^{p}\sum_{c_6=1}^{p}\frac{o_{c_3b}\{o_{c_2b}o_{c_6c_5}x_{c_1c_6}+o_{c_6b}o_{c_2c_5}x_{c_1c_6}\}}{\ell_{c_5}}\Big\} \\ &= \{\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\mathbf{Diag}\big(\sum_{b\neq 1}\frac{\psi_b}{\ell_b-\ell_1},\,\ldots,\,\sum_{b\neq n}\frac{\psi_b}{\ell_b-\ell_n}\big)\boldsymbol{O}_1'\}_{ic_3} \\ &\quad +\{\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\mathbf{Diag}\big(\frac{\psi_1}{\ell_1},\,\ldots,\,\frac{\psi_n}{\ell_n}\big)\boldsymbol{O}_1'\boldsymbol{O}_2\boldsymbol{O}_2'\}_{ic_3} \\ &\quad +\{\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_2\boldsymbol{O}_2'\}_{ic_3}\mathrm{Tr}\,(\boldsymbol{O}_1'\mathbf{Diag}\big(\frac{\psi_1}{\ell_1},\,\ldots,\,\frac{\psi_n}{\ell_n}\big)\boldsymbol{O}_1') \\ &= \{\boldsymbol{O}_1\mathbf{Diag}\big(\sum_{b\neq 1}\frac{\ell_1\psi_b}{\ell_b-\ell_1},\,\ldots,\,\sum_{b\neq n}\frac{\ell_n\psi_b}{\ell_b-\ell_n}\big)\boldsymbol{O}_1'\}_{ic_3}. \end{aligned}$$

□

Verification of $T_{ic_3}^{(13)}$:

$$\begin{aligned} T_{ic_3}^{(13)} &= 2\sum_{c_1,\,c_5,\,m=1}^{n}\sum_{c_2=1}^{p} x_{c_1 i}o_{c_2c_5}o_{c_3c_5}\frac{\partial\psi_{c_5}}{\partial\ell_m}\sum_{c_6=1}^{p} o_{c_6m}x_{c_1c_6}o_{c_2m} \\ &= 2\sum_{c_1,\,c_5}^{n} x_{c_1 i}o_{c_3c_5}\frac{\partial\psi_{c_5}}{\partial\ell_{c_5}}o_{c_6c_5}x_{c_1c_6} \\ &= \{\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\mathbf{Diag}\big(2\frac{\partial\psi_1}{\partial\ell_1},\,\ldots,\,2\frac{\partial\psi_n}{\partial\ell_n}\big)\boldsymbol{O}_1'\}_{ic_3} \\ &= \{\boldsymbol{O}_1\mathbf{Diag}\big(2\ell_1\frac{\partial\psi_1}{\partial\ell_1},\,\ldots,\,2\ell_n\frac{\partial\psi_n}{\partial\ell_n}\big)\boldsymbol{O}_1'\}_{ic_3}. \end{aligned}$$

□

Verification of $T^{(21)}_{c_4c_3}$:

$$
\begin{aligned}
T^{(21)}_{c_4c_3} &= \sum_{c_1,\,c_5=1}^{n}\sum_{c_2,\,i=1}^{p} x_{c_1i}\tilde{f}_{c_2c_3}o_{ic_5}\frac{\psi_{c_5}}{\ell_{c_5}}\Big\{\sum_{b\neq c_5}^{n}\sum_{c_6=1}^{p}\frac{o_{c_4b}\{o_{c_2b}o_{c_6c_5}x_{c_1c_6}+o_{c_6b}o_{c_2c_5}x_{c_1c_6}\}}{\ell_{c_5}-\ell_b} \\
&\qquad + \sum_{b=n+1}^{p}\sum_{c_6=1}^{p}\frac{o_{c_4b}\{o_{c_2b}o_{c_6c_5}x_{c_1c_6}+o_{c_6b}o_{c_2c_5}x_{c_1c_6}\}}{\ell_{c_5}}\Big\} \\
&= \sum_{c_5=1}^{n}\sum_{b\neq c_5}^{n} o_{c_4b}o_{c_2b}\tilde{f}_{c_2c_3}\frac{\psi_{c_5}}{\ell_{c_5}-\ell_b} + \sum_{c_5=1}^{n}\sum_{b=n+1}^{p} o_{c_4b}o_{c_2b}\tilde{f}_{c_2c_3}\frac{\psi_{c_5}}{\ell_{c_5}} \\
&= \{\boldsymbol{O}_1\mathbf{Diag}\big(\sum_{b\neq 1}\frac{\psi_b}{\ell_b-\ell_1},\,\ldots,\,\sum_{b\neq n}\frac{\psi_b}{\ell_b-\ell_n}\big)\boldsymbol{O}_1'\widetilde{\boldsymbol{F}}\}_{c_4c_3} + \{\boldsymbol{O}_2\boldsymbol{O}_2'\widetilde{\boldsymbol{F}}\}_{c_4c_3}\sum_{c_5=1}^{n}\frac{\psi_{c_5}}{\ell_{c_5}} \\
&= \{\boldsymbol{O}_1\mathbf{Diag}\big(\sum_{b\neq 1}\frac{\psi_1\psi_b}{\ell_b-\ell_1},\,\ldots,\,\sum_{b\neq n}\frac{\psi_1\psi_b}{\ell_b-\ell_n}\big)\boldsymbol{O}_1'\}_{c_4c_3}.
\end{aligned}
$$

The second equality can be seen as follows: Summing over c_6 and i from 1 to p and c_1 over from 1 to n, we can see that $\sum_{c_5,\,i=1}^{p}\sum_{c_1=1}^{n} o_{c_6c_5}x_{c_1c_6}x_{c_1i}o_{ic_5} = \ell_{c_5}$ and $\sum_{c_5,\,i=1}^{p}\sum_{c_1=1}^{n} o_{c_6b}x_{c_1c_6}x_{c_1i}o_{ic_5} = \ell_{c_5} = \delta_{bc_5}\ell_{c_5} = 0$ for $c_5 \neq b$. □

Verification of $T^{(22)}_{c_4c_3}$:

$$
\begin{aligned}
T^{(22)}_{c_4c_3} &= \sum_{c_1,\,c_5=1}^{n}\sum_{c_2,\,i=1}^{p} x_{c_1i}\tilde{f}_{c_2c_3}o_{c_4c_5}\frac{\psi_{c_5}}{\ell_{c_5}}\Big\{\sum_{b\neq c_5}^{n}\sum_{c_6=1}^{p}\frac{o_{ib}\{o_{c_2b}o_{c_6c_5}x_{c_1c_6}+o_{c_6b}o_{c_2c_5}x_{c_1c_6}\}}{\ell_{c_5}-\ell_b} \\
&\qquad + \sum_{b=n+1}^{p}\sum_{c_6=1}^{p}\frac{o_{ib}\{o_{c_2b}o_{c_6c_5}x_{c_1c_6}+o_{c_6b}o_{c_2c_5}x_{c_1c_6}\}}{\ell_{c_5}}\Big\} \\
&= \{\boldsymbol{O}_1\mathbf{Diag}\big(\sum_{b\neq 1}\frac{\ell_b\psi_1}{(\ell_1-\ell_b)\ell_1},\,\ldots,\,\sum_{b\neq n}\frac{\ell_b\psi_n}{(\ell_n-\ell_b)\ell_n}\big)\boldsymbol{O}_1'\widetilde{\boldsymbol{F}}\}_{c_4c_3} \\
&= \{\boldsymbol{O}_1\mathbf{Diag}\big(\sum_{b\neq 1}\frac{\ell_b\psi_1^2}{(\ell_1-\ell_b)\ell_1},\,\ldots,\,\sum_{b\neq n}\frac{\ell_b\psi_n^2}{(\ell_n-\ell_b)\ell_n}\big)\boldsymbol{O}_1'\}_{c_4c_3}.
\end{aligned}
$$

The second equality can be seen as follows: Summing over c_6 and i from 1 to p ano over c_1 from 1 to n, we can see that $\sum_{c_6,\,i=1}^{p}\sum_{c_1=1}^{n} o_{c_6c_5}x_{c_1c_6}x_{c_1i}o_{ib} = \{\boldsymbol{O}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_1\}_{c_5b} = \delta_{c_5b}\ell_b = 0$ for $c_5 \neq b$, and $\sum_{c_6,\,i=1}^{p}\sum_{c_1=1}^{n} o_{c_6b}x_{c_1c_6}x_{c_1i}o_{ib} = \{\boldsymbol{O}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{O}_2\}_{bb} = 0$ for $b = n+1,\,\ldots,\,p$. □

Verification of $T^{(23)}_{c_4c_3}$:

$$\begin{aligned}
T^{(23)}_{c_4c_3} &= 2\sum_{c_1,\,c_5,\,m=1}^{n}\sum_{c_2,\,i=1}^{p} x_{c_1i}\tilde{f}_{c_2c_3}o_{c_4c_5}o_{ic_5}\frac{\partial}{\partial\ell_m}\left(\frac{\psi_{c_5}}{\ell_{c_5}}\right)\sum_{c_6=1}^{p} o_{c_6m}x_{c_1c_6}o_{c_2m} \\
&= 2\sum_{c_5=1}^{n}\sum_{c_2=1}^{p}\ell_{c_5}\tilde{f}_{c_2c_3}o_{c_4c_5}o_{c_2c_5}\frac{\partial}{\partial\ell_{c_5}}\left(\frac{\psi_{c_5}}{\ell_{c_5}}\right) \\
&= \{\boldsymbol{O}_1\mathbf{Diag}\big(2\ell_1\frac{\partial}{\partial\ell_1}\left(\frac{\psi_1}{\ell_1}\right),\,\ldots,\,2\ell_n\frac{\partial}{\partial\ell_n}\left(\frac{\psi_n}{\ell_n}\right)\big)\boldsymbol{O}_1'\widetilde{\boldsymbol{F}}\}_{c_4c_3} \\
&= \{\boldsymbol{O}_1\mathbf{Diag}\big(2\ell_1\psi_1\frac{\partial}{\partial\ell_1}\left(\frac{\psi_1}{\ell_1}\right),\,\ldots,\,2\ell_n\psi_n\frac{\partial}{\partial\ell_n}\left(\frac{\psi_n}{\ell_n}\right)\big)\boldsymbol{O}_1'\}_{c_4c_3}.
\end{aligned}$$

□

A.3 Calculation for Proposition 2.2

Put $a = 1/(p+n+1)$ and apply Theorem 2.4 with $\varphi_k = a(\ell_k + t/\mathrm{Tr}\,\boldsymbol{S}^+)\,(k = 1,\,2,\,\ldots,\,n)$. Then we have, for $k = 1,\,2,\ldots,\,n$,

$$\begin{aligned}
\varphi^{(1)}_k &= \frac{(p-n-1)a^2}{\ell_k}\big(\ell_k+\frac{t}{\mathrm{Tr}\,\boldsymbol{S}^+}\big)^2 + 4a^2\big(\ell_k+\frac{t}{\mathrm{Tr}\,\boldsymbol{S}^+}\big)\big(1+\frac{t}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\big) \\
&\quad +2a^2(n-1)\big(\ell_k+\frac{t}{\mathrm{Tr}\,\boldsymbol{S}^+}\big) \\
&= a\ell_k + 2a^2\big(\frac{p}{\mathrm{Tr}\,\boldsymbol{S}^+}+\frac{2}{\ell_k(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\big)t + a^2\big(\frac{p-n-1}{\ell_k(\mathrm{Tr}\,\boldsymbol{S}^+)^2}+\frac{4}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{S}^+)^3}\big)t^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&R(\widehat{\boldsymbol{\Sigma}}_{\mathrm{HF}},\,\boldsymbol{\Sigma}) - R(\widehat{\boldsymbol{\Sigma}}_a,\,\boldsymbol{\Sigma}) \\
&= \mathbb{E}\Big[\sum_{k=1}^{n}\Big\{(p-n-1)\big\{2a^2\big(\frac{p}{\ell_k\mathrm{Tr}\,\boldsymbol{S}^+}+\frac{2}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\big)t + a^2\big(\frac{p-n-1}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{S}^+)^2}+\frac{4}{\ell_k^3(\mathrm{Tr}\,\boldsymbol{S}^+)^3}\big)t^2 \\
&\quad -\frac{2a}{\ell_k(\mathrm{Tr}\,\boldsymbol{S}^+)}t\big\} + 2\big\{2a^2\big(\frac{p}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{S}^+)^2}+\frac{4}{\ell_k^3(\mathrm{Tr}\,\boldsymbol{S}^+)^3}-\frac{2}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\big)t + a^2\big(\frac{2(p-n-1)}{\ell_k^3(\mathrm{Tr}\,\boldsymbol{S}^+)^3} \\
&\quad -\frac{p-n-1}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{S}^+)^2}+\frac{12}{\ell_k^4(\mathrm{Tr}\,\boldsymbol{S}^+)^4}-\frac{8}{\ell_k^3(\mathrm{Tr}\,\boldsymbol{S}^+)^3}\big)t^2 - \frac{2a}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{S}^+)^2}t\big\} \\
&\quad +\frac{4a^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\sum_{b\neq k}^{n}\frac{\ell_k^{-1}-\ell_b^{-1}}{\ell_k-\ell_b}\cdot t + \big(\frac{a^2(p-n-1)}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\sum_{b\neq k}^{n}\frac{\ell_k^{-1}-\ell_b^{-1}}{\ell_k-\ell_b}+\frac{4a^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^3}\sum_{b\neq k}^{n}\frac{\ell_k^{-2}-\ell_b^{-2}}{\ell_k-\ell_b}\big)t^2\Big\}\Big].
\end{aligned}$$

But we have

$$\begin{aligned}\sum_{k=1}^{n}\sum_{b\neq k}^{n}\frac{\ell_k^{-1}-\ell_b^{-1}}{\ell_k-\ell_b} &= \sum_{k=1}^{n}\sum_{b\neq k}^{n}\frac{\ell_b-\ell_k}{\ell_K\ell_b(\ell_k-\ell_b)} = -\sum_{k=1}^{n}\sum_{b\neq k}^{n}\frac{1}{\ell_k\ell_b} = -\sum_{k=1}^{n}\frac{1}{\ell_k}\sum_{b=1}^{n}\frac{1}{\ell_b}+\sum_{k=1}^{n}\frac{1}{\ell_k^2}\\ &= -(\mathrm{Tr}\,\boldsymbol{S}^+)^2+\mathrm{Tr}\,(\boldsymbol{S}^+)^2\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n}\sum_{b\neq k}^{n}\frac{\ell_k^{-2}-\ell_b^{-2}}{\ell_k-\ell_b} &= \sum_{k=1}^{n}\sum_{b\neq k}^{n}\frac{\ell_b-\ell_k}{\ell_k^2\ell_b^2(\ell_k-\ell_b)} = -\sum_{k=1}^{n}\sum_{b\neq k}^{n}\frac{\ell_k+\ell_b}{\ell_k^2\ell_b^2}\\ &= -\sum_{k=1}^{n}\frac{1}{\ell_k}\sum_{b=1}^{n}\frac{1}{\ell_b^2}+\sum_{k=1}^{n}\frac{1}{\ell_k^3}-\sum_{k=1}^{n}\frac{1}{\ell_k^2}\sum_{b=1}^{n}\frac{1}{\ell_b}+\sum_{k=1}^{n}\frac{1}{\ell_k^3}\\ &= -2(\mathrm{Tr}\,\boldsymbol{S}^+)\mathrm{Tr}\,(\boldsymbol{S}^+)^2+2\mathrm{Tr}\,(\boldsymbol{S}^+)^3.\end{aligned}$$

Therefore we have

$$\begin{aligned}&R(\widehat{\boldsymbol{\Sigma}}_{\mathrm{HF}},\,\boldsymbol{\Sigma})-R(\widehat{\boldsymbol{\Sigma}}_a,\,\boldsymbol{\Sigma})\\ &=\mathbb{E}\Big[(p-n-1)\big\{2a^2\big(p+\frac{2\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\big)t+a^2\big(\frac{(p-n-1)\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}+\frac{4\mathrm{Tr}\,(\boldsymbol{S}^+)^3}{(\mathrm{Tr}\,\boldsymbol{S}^+)^3}\big)t^2\\ &\quad -2at\big\}+2\big\{2a^2\big(\frac{p\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}+\frac{4\mathrm{Tr}\,(\boldsymbol{S}^+)^3}{(\mathrm{Tr}\,\boldsymbol{S}^+)^3}-\frac{2\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\big)t+a^2\big(\frac{2(p-n-1)\mathrm{Tr}\,(\boldsymbol{S}^+)^3}{(\mathrm{Tr}\,\boldsymbol{S}^+)^3}\\ &\quad -\frac{(p-n-1)\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}+\frac{12\mathrm{Tr}\,(\boldsymbol{S}^+)^4}{(\mathrm{Tr}\,\boldsymbol{S}^+)^4}-\frac{8\mathrm{Tr}\,(\boldsymbol{S}^+)^3}{(\mathrm{Tr}\,\boldsymbol{S}^+)^3}\big)t^2-\frac{2a\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}t\big\}\\ &\quad +\frac{4a^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}(-(\mathrm{Tr}\,\boldsymbol{S}^+)^2+\mathrm{Tr}\,(\boldsymbol{S}^+)^2)\cdot t+\big(\frac{a^2(p-n-1)}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}(-(\mathrm{Tr}\,\boldsymbol{S}^+)^2+\mathrm{Tr}\,(\boldsymbol{S}^+)^2)\\ &\quad +\frac{4a^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^3}(-2(\mathrm{Tr}\,\boldsymbol{S}^+)\mathrm{Tr}\,(\boldsymbol{S}^+)^2+2\mathrm{Tr}\,(\boldsymbol{S}^+)^3)\big)t^2\Big]\\ &< a\mathbb{E}\Big[(p-n-1)\big\{2a\big(p+\frac{2\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}\big)t+a\frac{(p-n+3)\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}t^2-2t\big\}\\ &\quad +2\big\{2a\frac{(p+2)\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}t+a\frac{(p-n+3)\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}t^2-\frac{2\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}t\big\}\Big].\end{aligned}$$

Since the coefficients of $\{\mathrm{Tr}\,(\boldsymbol{S}^+)^2/(\mathrm{Tr}\,\boldsymbol{S}^+)^2\}t$ is evaluated as

$$2a(p-n-1)+4a(p+1)-4=2a(p-n-1)-4a(n-1)<2a(p-n-1),$$

we have

$$\{2a(p-n-1)+4a(p+1)-4\}\frac{\mathrm{Tr}\,(\boldsymbol{S}^+)^2}{(\mathrm{Tr}\,\boldsymbol{S}^+)^2}t < 2a(p-n-1)t,$$

from which it follows that

$$R(\widehat{\boldsymbol{\Sigma}}_{\mathrm{HF}},\,\boldsymbol{\Sigma}) - R(\widehat{\boldsymbol{\Sigma}}_a,\,\boldsymbol{\Sigma}) < a^2\big\{(p-n+1)(p-n+3)t^2 - 2(n-1)(p-n-1)t\big\}.$$

This completes the proof. □

A.4 Calculation for the proof of Theorem 3.2

Verification of A_4 **:**

$$\begin{aligned} A_4 &= \sum_{c_1,\,c_3,\,c_4=1}^{n}\sum_{c_2=1}^{p} z_{c_1c_2}\psi_{c_3}\overline{u}_{jc_3}\bigg\{\sum_{b\neq c_3}^{n}\sum_{c_4=1}^{p}\frac{u_{c_2b}\overline{u}_{c_4b}u_{ic_3}\overline{z}_{c_1c_4}}{\ell_{c_3}-\ell_b} + \sum_{b=n+1}^{p}\sum_{c_4=1}^{p}\frac{u_{c_2b}\overline{u}_{c_4b}u_{ic_3}\overline{z}_{c_1c_4}}{\ell_{c_3}}\bigg\} \\ &= \sum_{c_3=1}^{n} u_{ic_3}\overline{u}_{jc_3}\bigg\{\sum_{b\neq c_3}^{n}\frac{\psi_{c_3}}{\ell_{c_3}-\ell_b}\{\boldsymbol{U}_1'\boldsymbol{Z}'\overline{\boldsymbol{Z}}\,\overline{\boldsymbol{U}}_1\}_{bb} + \sum_{b=n+1}^{p}\frac{\psi_{c_3}}{\ell_{c_3}}\{\boldsymbol{U}_2'\boldsymbol{Z}'\overline{\boldsymbol{Z}}\,\overline{\boldsymbol{U}}_2\}_{bb}\bigg\} \\ &= \sum_{c_3=1}^{n} u_{ic_3}\overline{u}_{jc_3}\sum_{b\neq c_3}^{n}\frac{\ell_b\psi_{c_3}}{\ell_{c_3}-\ell_b}. \end{aligned}$$

□

Verification of A_5 **:**

$$\begin{aligned} A_5 &= \sum_{c_1,\,c_3,\,c_4=1}^{n}\sum_{c_2=1}^{p} z_{c_1c_2}\psi_{c_3}u_{c_2c_3}\bigg\{\sum_{b\neq c_3}^{n}\frac{\overline{u}_{jb}u_{ib}\overline{u}_{c_4c_3}\overline{z}_{c_1c_4}}{\ell_{c_3}-\ell_b} + \sum_{b=n+1}^{p}\frac{\overline{u}_{jb}u_{ib}\overline{u}_{c_4c_3}\overline{z}_{c_1c_4}}{\ell_{c_3}}\bigg\} \\ &= \sum_{b=1}^{n}\sum_{c_3\neq b}^{n}\frac{u_{ib}\overline{u}_{jb}\psi_{c_3}}{\ell_{c_3}-\ell_b}\{\boldsymbol{U}_1'\boldsymbol{Z}'\overline{\boldsymbol{Z}}\,\overline{\boldsymbol{U}}_1\}_{c_3c_3} + \sum_{b=n+1}^{p}\sum_{c_3=1}^{n}\{\boldsymbol{U}_1'\boldsymbol{Z}'\overline{\boldsymbol{Z}}\,\overline{\boldsymbol{U}}_1\}_{c_3c_3}\frac{u_{ib}\overline{u}_{jb}\psi_{c_3}}{\ell_{c_3}} \\ &= \sum_{b=1}^{n} u_{ib}\overline{u}_{jb}\sum_{c_3\neq b}\frac{\ell_{c_3}\psi_{c_3}}{\ell_{c_3}-\ell_b} + \sum_{b=n+1}^{p} u_{ib}\overline{u}_{jb}\sum_{c_3=1}^{n}\psi_{c_3}. \end{aligned}$$

□

Verification of A_6 :

$$
\begin{aligned}
A_6 &= \sum_{c_1,\,c_3,\,c_4,\,m=1}^{n} \sum_{c_2=1}^{p} z_{c_1c_2} u_{c_2c_3} \overline{u}_{jc_3} \frac{\partial \psi_{c_3}}{\partial \ell_m} \overline{u}_{c_4m} u_{im} \overline{z}_{c_1c_4} \\
&= \sum_{c_3,\,m=1}^{n} u_{im} \overline{u}_{jc_3} \frac{\partial \psi_{c_3}}{\partial \ell_m} \{\boldsymbol{U}_1' \boldsymbol{Z}' \overline{\boldsymbol{Z}}\, \overline{\boldsymbol{U}}_1\}_{c_3m} = \sum_{c_3=1}^{n} u_{ic_3} \overline{u}_{jc_3} \ell_{c_3} \frac{\partial \psi_{c_3}}{\partial \ell_{c_3}}.
\end{aligned}
$$

□

A.5 Calculation for Theorem 3.3

Verification of $T^{(31)}_{c_4c_2}$:

$$
\begin{aligned}
T^{(31)}_{c_4c_2} &= \sum_{c_1,\,c_5=1}^{n} \sum_{i=1}^{p} z_{c_1c_2} \psi_{c_5} \bar{u}_{ic_5} \left\{ \sum_{b\neq c_5}^{n} \sum_{c_6=1}^{p} \frac{u_{c_4b} \bar{u}_{c_6b} u_{ic_5} \bar{z}_{c_1c_6}}{\ell_{c_5} - \ell_b} + \sum_{b=n+1}^{p} \sum_{c_6=1}^{p} \frac{u_{c_4b} \bar{u}_{c_6b} u_{ic_5} \bar{z}_{c_1c_6}}{\ell_{c_5}} \right\} \\
&= \{\boldsymbol{U}_1 \mathbf{Diag}\big(\sum_{b\neq 1} \frac{\psi_b}{\ell_b - \ell_1},\, \ldots,\, \sum_{b\neq n} \frac{\psi_b}{\ell_b - \ell_n}\big) \boldsymbol{U}_1^* \boldsymbol{Z}^* \boldsymbol{Z}\}_{c_4c_2} + \{\boldsymbol{U}_2 \boldsymbol{U}_2^* \boldsymbol{Z}^* \boldsymbol{Z}\}_{c_4c_2} \sum_{c_5}^{n} \frac{\psi_{c_5}}{\ell_{c_5}} \\
&= \{\boldsymbol{U}_1 \mathbf{Diag}\big(\sum_{b\neq 1} \frac{\ell_1 \psi_b}{\ell_b - \ell_1},\, \ldots,\, \sum_{b\neq n} \frac{\ell_n \psi_b}{\ell_b - \ell_n}\big) \boldsymbol{U}_1^*\}_{c_4c_2}.
\end{aligned}
$$

□

Verification of $T^{(32)}_{c_4c_2}$:

$$
\begin{aligned}
T^{(32)}_{c_4c_2} &= \sum_{c_1,\,c_5=1}^{n} \sum_{i=1}^{p} z_{c_1c_2} \psi_{c_5} \bar{u}_{c_4c_5} \left\{ \sum_{b\neq c_5}^{n} \sum_{c_6=1}^{p} \frac{\bar{u}_{ib} u_{ib} \bar{u}_{c_6c_5} z_{c_1c_6}}{\ell_{c_5} - \ell_b} + \sum_{b=n+1}^{p} \sum_{c_6=1}^{p} \frac{\bar{u}_{ib} u_{ib} \bar{u}_{c_6c_5} z_{c_1c_6}}{\ell_{c_5}} \right\} \\
&= \{\boldsymbol{U}_1 \mathbf{Diag}\big(\sum_{b\neq 1} \frac{\psi_1}{\ell_1 - \ell_b},\, \ldots,\, \sum_{b\neq n} \frac{\psi_n}{\ell_n - \ell_b}\big) \boldsymbol{U}_1^* \boldsymbol{Z}^* \boldsymbol{Z}\}_{c_4c_2} \\
&\quad + (p-n)\{\boldsymbol{U}_1 \mathbf{Diag}\big(\frac{\psi_1}{\ell_1},\, \ldots,\, \frac{\psi_n}{\ell_n}\big) \boldsymbol{U}_1^* \boldsymbol{Z}^* \boldsymbol{Z}\}_{c_4c_2} \\
&= \{\boldsymbol{U}_1 \mathbf{Diag}\big(\sum_{b\neq 1} \frac{\ell_1 \psi_1}{\ell_1 - \ell_b},\, \ldots,\, \sum_{b\neq n} \frac{\ell_n \psi_n}{\ell_n - \ell_b}\big) \boldsymbol{U}_1^*\}_{c_4c_2} \\
&\quad + (p-n)\{\boldsymbol{U}_1 \mathbf{Diag}\big(\psi_1,\, \ldots,\, \psi_n\big) \boldsymbol{U}_1^*\}_{c_4c_2}.
\end{aligned}
$$

$\square$

Verification of $T^{(33)}_{c_4c_2}$:

$$\begin{aligned}
T^{(33)}_{c_4c_2} &= \sum_{c_1,\, c_5,\, m=1}^{n}\sum_{i=1}^{p} z_{c_1c_2}u_{c_4c_5}\bar{u}_{ic_5}\frac{\partial\psi_{c_5}}{\partial\ell_m}\sum_{c_6=1}^{p}\bar{u}_{c_6m}\bar{z}_{c_1c_6}u_{im}\\
&= \sum_{c_1,\, c_5=1}^{n}\sum_{c_6=1}^{p} z_{c_1c_2}u_{c_4c_5}\frac{\partial\psi_{c_5}}{\partial\ell_{c_5}}\bar{u}_{c_6c_5}\bar{z}_{c_1c_6}\\
&= \{\boldsymbol{U}_1\mathbf{Diag}(\frac{\partial\psi_1}{\partial\ell_1},\, \ldots,\, \frac{\partial\psi_n}{\partial\ell_n})\boldsymbol{U}_1^*\boldsymbol{Z}^*\boldsymbol{Z}\}_{c_4c_2}\\
&= \{\boldsymbol{U}_1\mathbf{Diag}(\ell_1\frac{\partial\psi_1}{\partial\ell_1},\, \ldots,\, \ell_n\frac{\partial\psi_n}{\partial\ell_n})\boldsymbol{U}_1^*\}_{c_4c_2}.
\end{aligned}$$

$\square$

Verification of $T^{(41)}_{c_4c_3}$:

$$\begin{aligned}
T^{(41)}_{c_4c_3} &= \sum_{c_1,\, c_5=1}^{n}\sum_{c_2,\, i=1}^{p} z_{c_1c_2}\tilde{f}_{c_4i}\bar{u}_{c_3c_5}\frac{\psi_{c_5}}{\ell_{c_5}}\Bigg\{\sum_{b\neq c_5}^{n}\sum_{c_6=1}^{p}\frac{u_{c_2b}\bar{u}_{c_6b}u_{ic_5}\bar{z}_{c_1c_6}}{\ell_{c_5}-\ell_b}+\sum_{b=n+1}^{p}\sum_{c_6=1}^{p}\frac{u_{c_2b}\bar{u}_{c_6b}u_{ic_5}\bar{z}_{c_1c_6}}{\ell_{c_5}}\Bigg\}\\
&= \sum_{c_5=1}^{n}\sum_{i=1}^{p}\sum_{b\neq c_5}^{n}\tilde{f}_{c_4i}\bar{u}_{c_3c_5}u_{ic_5}\frac{\psi_{c_5}}{\ell_{c_4}}\frac{\{\boldsymbol{U}_1'\boldsymbol{Z}'\overline{\boldsymbol{Z}}\,\overline{\boldsymbol{U}}_1\}_{bb}}{\ell_{c_5}-\ell_b}+\sum_{b=n+1}^{p}\sum_{c_5=1}^{n}\sum_{i=1}^{p}\tilde{f}_{c_4i}\bar{u}_{c_3c_5}u_{ic_5}\frac{\psi_{c_5}}{\ell_{c_4}}\frac{\{\boldsymbol{U}_2'\boldsymbol{Z}'\overline{\boldsymbol{Z}}\,\overline{\boldsymbol{U}}_2\}_{bb}}{\ell_{c_5}}\Bigg\}\\
&= \{\widetilde{\boldsymbol{F}}\boldsymbol{U}_1\mathbf{Diag}(\sum_{b\neq 1}\frac{\ell_b\psi_1}{\ell_1(\ell_1-\ell_b)},\, \ldots,\, \sum_{b\neq n}\frac{\ell_b\psi_n}{\ell_n(\ell_n-\ell_b)})\boldsymbol{U}_1^*\}_{c_4c_3}\\
&= \{\boldsymbol{U}_1\mathbf{Diag}(\sum_{b\neq 1}\frac{\ell_b\psi_1^2}{\ell_1(\ell_1-\ell_b)},\, \ldots,\, \sum_{b\neq n}\frac{\ell_b\psi_n^2}{\ell_n(\ell_n-\ell_b)})\boldsymbol{U}_1^*\}_{c_4c_3}.
\end{aligned}$$

$\square$

Verification of $T^{(42)}_{c_4c_3}$:

$$\begin{aligned}
T^{(42)}_{c_4c_3} &= \sum_{c_1,\, c_5=1}^{n}\sum_{c_2,\, i=1}^{p} z_{c_1c_2}\tilde{f}_{c_4i}u_{c_2c_5}\frac{\psi_{c_5}}{\ell_{c_5}}\Bigg\{\sum_{b\neq c_5}^{n}\sum_{c_6=1}^{p}\frac{\bar{u}_{c_3b}u_{ib}\bar{u}_{c_6c_5}\bar{z}_{c_1c_6}}{\ell_{c_5}-\ell_b}+\sum_{b=n+1}^{p}\sum_{c_6=1}^{p}\frac{\bar{u}_{c_3b}u_{ib}\bar{u}_{c_6c_5}\bar{z}_{c_1c_6}}{\ell_{c_5}}\Bigg\}\\
&= \sum_{c_5=1}^{n}\sum_{i=1}^{p}\sum_{b\neq c_5}^{n}\tilde{f}_{c_4i}u_{ib}\bar{u}_{c_3b}\frac{\{\boldsymbol{U}_1^*\boldsymbol{Z}^*\boldsymbol{Z}\boldsymbol{U}_1\}_{c_5c_5}\psi_{c_5}}{\ell_{c_5}-\ell_b}+\sum_{b=n+1}^{p}\sum_{c_5=1}^{n}\sum_{i=1}^{p}\tilde{f}_{c_4i}u_{ib}\bar{u}_{c_3b}\frac{\{\boldsymbol{U}_1^*\boldsymbol{Z}^*\boldsymbol{Z}\boldsymbol{U}_1\}_{c_5c_5}\psi_{c_5}}{\ell_{c_5}^2}\Bigg\}\\
&= \{\widetilde{\boldsymbol{F}}\boldsymbol{U}_1\mathbf{Diag}(\sum_{b\neq 1}\frac{\psi_b}{\ell_b(\ell_b-\ell_1)},\, \ldots,\, \sum_{b\neq n}\frac{\psi_b}{\ell_b(\ell_b-\ell_n)})\boldsymbol{U}_1^*\}_{c_4c_3}+\{\widetilde{\boldsymbol{F}}\boldsymbol{U}_2\boldsymbol{U}_2^*\}_{c_4c_3}\sum_{c_5=1}^{n}\frac{\psi_{c_5}}{\ell_{c_5}}\\
&= \{\boldsymbol{U}_1\mathbf{Diag}(\sum_{b\neq 1}\frac{\psi_1\psi_b}{\ell_b(\ell_b-\ell_1)},\, \ldots,\, \sum_{b\neq n}\frac{\psi_n\psi_b}{\ell_b(\ell_b-\ell_n)})\boldsymbol{U}_1^*\}_{c_4c_3}.
\end{aligned}$$

□

Verification of $T^{(43)}_{c_4c_3}$:

$$\begin{aligned} T^{(43)}_{c_4c_3} &= \sum_{c_1,\,c_5,\,m=1}^{n}\sum_{c_2,\,i=1}^{p} z_{c_1c_2}\tilde{f}_{c_4i}u_{c_2c_5}\bar{u}_{c_3c_5}\frac{\partial}{\partial \ell_m}\left(\frac{\psi_{c_5}}{\ell_{c_5}}\right)\sum_{c_6=1}^{p}\bar{u}_{c_6m}\bar{z}_{c_1c_6}u_{im} \\ &= \sum_{c_1,\,c_5=1}^{n}\sum_{c_6=1}^{p}\tilde{f}_{c_4i}u_{ic_5}\bar{u}_{c_3c_5}\ell_{c_5}\frac{\partial}{\partial \ell_{c_5}}\left(\frac{\psi_{c_5}}{\ell_{c_5}}\right) \\ &= \{\widetilde{\boldsymbol{F}}\boldsymbol{U}_1\mathbf{Diag}\big(\ell_1\frac{\partial}{\partial \ell_1}\left(\frac{\psi_1}{\ell_1}\right),\,\ldots,\,\ell_n\frac{\partial}{\partial \ell_n}\left(\frac{\psi_n}{\ell_n}\right)\big)\boldsymbol{U}_1^*\boldsymbol{Z}^*\boldsymbol{Z}\}_{c_4c_3} \\ &= \{\boldsymbol{U}_1\mathbf{Diag}\big(\ell_1\psi_1\frac{\partial}{\partial \ell_1}\left(\frac{\psi_1}{\ell_1}\right),\,\ldots,\,\ell_n\psi_n\frac{\partial}{\partial \ell_n}\left(\frac{\psi_n}{\ell_n}\right)\big)\boldsymbol{U}_1^*\}_{c_4c_3}. \end{aligned}$$

□

A.6 Verification of the third equation in Lemma 4.2

$$\begin{aligned} \frac{\partial \bar{u}_{ak}}{\partial z_{ij}} &= \sum_{b=1,\,c_1=1}^{p}\bar{u}_{ab}u_{c_1b}\frac{\partial \bar{u}_{c_1k}}{\partial z_{ij}} \\ &= \sum_{b=1}^{n}\bar{u}_{ab}\{\boldsymbol{U}_1'(d\overline{\boldsymbol{U}}_1)\}_{bk}\left(\frac{\partial}{\partial z_{ij}}\right)+\sum_{b=n+1}^{p}\bar{u}_{ab}\{\boldsymbol{U}_2'(d\overline{\boldsymbol{U}}_1)\}_{bk}\left(\frac{\partial}{\partial z_{ij}}\right) \\ &= \sum_{b\neq k}^{n}\frac{\bar{u}_{ab}}{\ell_k-\ell_b}\{\boldsymbol{U}_1'\big((d\boldsymbol{Z}')\overline{\boldsymbol{Z}}+\boldsymbol{Z}'(d\overline{\boldsymbol{Z}})\big)\overline{\boldsymbol{U}}_1\}_{bk}\left(\frac{\partial}{\partial z_{ij}}\right) \\ &\qquad+\sum_{b=n+1}^{p}\frac{\bar{u}_{ab}}{\ell_k}\{\boldsymbol{U}_2'\big((d\boldsymbol{Z}')\overline{\boldsymbol{Z}}+\boldsymbol{Z}'(d\overline{\boldsymbol{Z}})\big)\overline{\boldsymbol{U}}_1\}_{bk}\left(\frac{\partial}{\partial z_{ij}}\right) \\ &= \sum_{b\neq k}^{n}\frac{\bar{u}_{ab}u_{c_1b}\bar{u}_{c_3k}}{\ell_k-\ell_b}\{(dz_{c_2c_1})\bar{z}_{c_2c_3}+z_{c_2c_1}(d\bar{z}_{c_2c_3})\}\left(\frac{\partial}{\partial z_{ij}}\right) \\ &\qquad+\sum_{b=n+1}^{p}\frac{\bar{u}_{ab}u_{c_1b}\bar{u}_{c_3k}}{\ell_k}\{(dz_{c_2c_1})\bar{z}_{c_2c_3}+z_{c_2c_1}(d\bar{z}_{c_2c_3})\}\left(\frac{\partial}{\partial z_{ij}}\right) \\ &= \sum_{b\neq k}^{n}\sum_{c_1=1}^{p}\frac{\bar{u}_{ab}u_{jb}\bar{u}_{c_1k}\bar{z}_{ic_1}}{\ell_k-\ell_b}+\sum_{b=n+1}^{p}\sum_{c_1=1}^{p}\frac{\bar{u}_{ab}u_{jb}\bar{u}_{c_1k}\bar{z}_{ic_1}}{\ell_k}. \end{aligned}$$

□

A.7 Calculation for Proposition 3.2

Put $a = 1/(p+n)$ and apply Theorem 2.4 with $\varphi_k = a(\ell_k + t/\mathrm{Tr}\,\boldsymbol{W}^+)$ $(k = 1,\, 2,\, \ldots,\, n)$. Then we have, for $k = 1,\, 2, \ldots,\, n$,

$$\begin{aligned}
\varphi_k^{(1)} &= \frac{(p-n)a^2}{\ell_k}\big(\ell_k + \frac{t}{\mathrm{Tr}\,\boldsymbol{W}^+}\big)^2 + 2a^2\big(\ell_k + \frac{t}{\mathrm{Tr}\,\boldsymbol{W}^+}\big)\big(1 + \frac{t}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\big) \\
&\quad +2a^2(n-1)\big(\ell_k + \frac{t}{\mathrm{Tr}\,\boldsymbol{W}^+}\big) \\
&= a\ell_k + 2a^2\big(\frac{p}{\mathrm{Tr}\,\boldsymbol{W}^+} + \frac{1}{\ell_k(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\big)t + a^2\big(\frac{p-n}{\ell_k(\mathrm{Tr}\,\boldsymbol{W}^+)^2} + \frac{2}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{W}^+)^3}\big)t^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&R(\widehat{\boldsymbol{\Sigma}}_{\mathrm{HF}},\, \boldsymbol{\Sigma}) - R(\widehat{\boldsymbol{\Sigma}}_a,\, \boldsymbol{\Sigma}) \\
&= \mathbb{E}\Big[\sum_{k=1}^n\Big\{(p-n)\big\{2a^2\big(\frac{p}{\ell_k\mathrm{Tr}\,\boldsymbol{W}^+} + \frac{1}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\big)t + a^2\big(\frac{p-n}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{W}^+)^2} + \frac{2}{\ell_k^3(\mathrm{Tr}\,\boldsymbol{W}^+)^3}\big)t^2 \\
&\quad -\frac{2a}{\ell_k(\mathrm{Tr}\,\boldsymbol{W}^+)}t\big\} + \big\{2a^2\big(\frac{p}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{W}^+)^2} + \frac{2}{\ell_k^3(\mathrm{Tr}\,\boldsymbol{W}^+)^3} - \frac{1}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\big)t + a^2\big(\frac{2(p-n)}{\ell_k^3(\mathrm{Tr}\,\boldsymbol{W}^+)^3} \\
&\quad -\frac{p-n}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{W}^+)^2} + \frac{6}{\ell_k^4(\mathrm{Tr}\,\boldsymbol{W}^+)^4} - \frac{4}{\ell_k^3(\mathrm{Tr}\,\boldsymbol{W}^+)^3}\big)t^2 - \frac{2a}{\ell_k^2(\mathrm{Tr}\,\boldsymbol{W}^+)^2}t\big\} \\
&\quad +\frac{2a^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\sum_{b\neq k}^n\frac{\ell_k^{-1}-\ell_b^{-1}}{\ell_k-\ell_b}\cdot t + \big(\frac{a^2(p-n)}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\sum_{b\neq k}^n\frac{\ell_k^{-1}-\ell_b^{-1}}{\ell_k-\ell_b} + \frac{2a^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^3}\sum_{b\neq k}^n\frac{\ell_k^{-2}-\ell_b^{-2}}{\ell_k-\ell_b}\big)t^2\Big\}\Big].
\end{aligned}$$

But we have

$$\begin{aligned}
\sum_{k=1}^n\sum_{b\neq k}^n\frac{\ell_k^{-1}-\ell_b^{-1}}{\ell_k-\ell_b} &= \sum_{k=1}^n\sum_{b\neq k}^n\frac{\ell_b-\ell_k}{\ell_K\ell_b(\ell_k-\ell_b)} = -\sum_{k=1}^n\sum_{b\neq k}^n\frac{1}{\ell_k\ell_b} = -\sum_{k=1}^n\frac{1}{\ell_k}\sum_{b=1}^n\frac{1}{\ell_b} + \sum_{k=1}^n\frac{1}{\ell_k^2} \\
&= -(\mathrm{Tr}\,\boldsymbol{W}^+)^2 + \mathrm{Tr}\,(\boldsymbol{W}^+)^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^n\sum_{b\neq k}^n\frac{\ell_k^{-2}-\ell_b^{-2}}{\ell_k-\ell_b} &= \sum_{k=1}^n\sum_{b\neq k}^n\frac{\ell_b-\ell_k}{\ell_k^2\ell_b^2(\ell_k-\ell_b)} = -\sum_{k=1}^n\sum_{b\neq k}^n\frac{\ell_k+\ell_b}{\ell_k^2\ell_b^2} \\
&= -\sum_{k=1}^n\frac{1}{\ell_k}\sum_{b=1}^n\frac{1}{\ell_b^2} + \sum_{k=1}^n\frac{1}{\ell_k^3} - \sum_{k=1}^n\frac{1}{\ell_k^2}\sum_{b=1}^n\frac{1}{\ell_b} + \sum_{k=1}^n\frac{1}{\ell_k^3} \\
&= -2(\mathrm{Tr}\,\boldsymbol{W}^+)\mathrm{Tr}\,(\boldsymbol{W}^+)^2 + 2\mathrm{Tr}\,(\boldsymbol{W}^+)^3.
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
&R(\widehat{\boldsymbol{\Sigma}}_{\mathrm{HF}},\, \boldsymbol{\Sigma}) - R(\widehat{\boldsymbol{\Sigma}}_a,\, \boldsymbol{\Sigma}) \\
&= \mathbb{E}\Big[(p-n)\big\{2a^2\big(p + \frac{\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\big)t + a^2\big(\frac{(p-n)\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2} + \frac{2\mathrm{Tr}\,(\boldsymbol{W}^+)^3}{(\mathrm{Tr}\,\boldsymbol{W}^+)^3}\big)t^2 \\
&\quad -2at\big\} + \big\{2a^2\big(\frac{p\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2} + \frac{2\mathrm{Tr}\,(\boldsymbol{W}^+)^3}{(\mathrm{Tr}\,\boldsymbol{W}^+)^3} - \frac{\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\big)t + a^2\big(\frac{2(p-n)\mathrm{Tr}\,(\boldsymbol{W}^+)^3}{(\mathrm{Tr}\,\boldsymbol{W}^+)^3} \\
&\quad -\frac{(p-n)\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2} + \frac{6\mathrm{Tr}\,(\boldsymbol{W}^+)^4}{(\mathrm{Tr}\,\boldsymbol{W}^+)^4} - \frac{4\mathrm{Tr}\,(\boldsymbol{W}^+)^3}{(\mathrm{Tr}\,\boldsymbol{W}^+)^3}\big)t^2 - \frac{2a\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}t\big\} \\
&\quad +\frac{2a^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}(-(\mathrm{Tr}\,\boldsymbol{W}^+)^2 + \mathrm{Tr}\,(\boldsymbol{W}^+)^2)\cdot t + \big(\frac{a^2(p-n)}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}(-(\mathrm{Tr}\,\boldsymbol{W}^+)^2 + \mathrm{Tr}\,(\boldsymbol{W}^+)^2) \\
&\quad +\frac{2a^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^3}(-2(\mathrm{Tr}\,\boldsymbol{W}^+)\mathrm{Tr}\,(\boldsymbol{W}^+)^2 + 2\mathrm{Tr}\,(\boldsymbol{W}^+)^3)\big)t^2\Big] \\
&< a\mathbb{E}\Big[(p-n)\big\{2a\big(p + \frac{\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}\big)t + a\frac{(p-n+2)\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}t^2 - 2t\big\} \\
&\quad +\big\{2a\frac{(p+1)\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}t + a\frac{(p-n+2)\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}t^2 - \frac{2\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}t\big\}\Big].
\end{aligned}
$$

Since the coefficients of $\{\mathrm{Tr}\,(\boldsymbol{W}^+)^2/(\mathrm{Tr}\,\boldsymbol{W}^+)^2\}t$ is evaluated as

$$
2a(p-n) + 2a(p+1) - 2 = 2a(p-n) - 2a(n-1) < 2a(p-n),
$$

we have

$$
\{2a(p-n) + 2a(p+1) - 2\}\frac{\mathrm{Tr}\,(\boldsymbol{W}^+)^2}{(\mathrm{Tr}\,\boldsymbol{W}^+)^2}t < 2a(p-n)t,
$$

from which it follows that

$$
R(\widehat{\boldsymbol{\Sigma}}_{\mathrm{HF}},\, \boldsymbol{\Sigma}) - R(\widehat{\boldsymbol{\Sigma}}_a,\, \boldsymbol{\Sigma}) < a^2\big\{(p-n+2)(p-n+1)t^2 - 2(n-1)(p-n)t\big\}.
$$

This completes the proof. $\square$