CALCULATING FUNDAMENTAL SURFACES BY HAND

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Abstract. We calculate fundamental surfaces by hand for the solid torus triangulated with 3 tetrahedra, the $(3,1)$-lens space with 3 tetrahedra and the 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ with 6 tetrahedra.

1. Introduction

The theory of normal surfaces was introduced by Kneser in [9] and developed by Haken in [4]. Haken introduced fundamental surfaces to obtain an algorithm to decide if a given knot is trivial or not. This theory gives important algorithms for 3-manifold topology. See, for example, [7] and [13]. It was also applied to problems other than algorithms. See [5] and [10], for example.

Normal surfaces in the figure eight complement with an ideal triangulation are studied very well. See [1], [8], [11] and [12]. In [3], E. A. Fominykh gives a complete description of the fundamental surfaces for infinite series of 3-manifolds including (p, q)-lens spaces $(p \geq 4)$ with respect to special kind of handle decompositions. His argumet is gemetrical. We calculate fundamental surfaces of some triangulated 3-manifolds algebraically by hand rather than by a computer. B. A. Burton made a computer program *Regina* which calculates all the vertex solutions of matching equations for triangulated 3-manifolds. See [2]. The authors don't know whether there is a software which calculates all the fundamental surfaces for general triangulated 3-manifolds or not. They expect that methods used in this paper give a hint on a good algorithm for computers. However, they are weak at computer programming.

Remark 1.1. For a given triangulation, we can form a handle decomposition by thickening 0-cells, 1-cells and 2-cells and thinning 3-cells to obtain 3-handles, 2-handles, 1-handles and 0-handles respectively. Thus the normal surface theory on triangulations can be regarded as Haken's original normal surface theory on a handle decomposition. However, triangulations are easier to treat by computer.

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Remark 1.2. In many cases, it is enough to obtain all the normal surfaces corresponding to vertex solutions rather than fundamental surfaces. See [7]. However, the authors think that fundamental surfaces are easier to calculate by hand than normal surfaces corresponding to vertex solutions.

In section 2, we calculate fundamental surfaces for a triangulated solid torus with 3-tetrahedra. We show the list of the fundamental surfaces and calculation in 5 important cases among all the 18 cases. In section 3, we show the whole of the calculation of all the fundamental surfaces for a triangulated 3-torus with 6-tetrahedra. In section 4, we give the list of all the fundamental surfaces for a triangulation of the $(2,1)$- and (3,1)-lens spaces with 2- and 3-tetrahedra respectively. We also show calculation of (possibly non-admissible) fundamental solutions of the matching equations in a single case. The authors gave up calculating the whole cases by hand because there are too many cases. In section 5, we give data on other triangulations of $(3,1)$- and $(4,1)$- lens spaces. In section 6, we introduce a method of showing non-triviality of the figure 8 knot, using Euler characteristics of normal surfaces rather than fundamental surfaces.

2. Solid torus

In this section, we calculate all the fundamental sufaces in a solid torus triangulated with 3 tetrahedra. A triangular prism can be decomposed into 3 tetrahedra $\tau_i = v_{i1}v_{i2}v_{i3}v_{i4}$ ($i = 1, 2$ and 3) as in Figure 1.

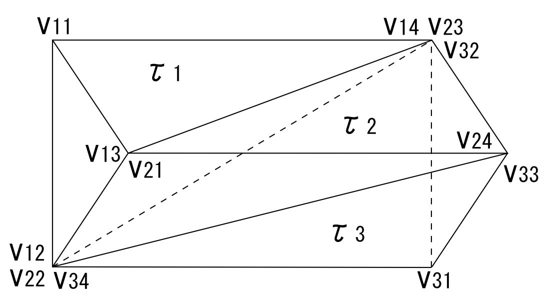

Figure 1. A triangulation of the solid torus

The faces $v_{12}v_{13}v_{14}$ and $v_{22}v_{21}v_{23}$ are glued so that the vertices are amalgamated in this order. The faces $v_{22}v_{23}v_{24}$ and $v_{34}v_{32}v_{33}$ are also attached. We obtain the solid torus W by identifying the bases $v_{11}v_{12}v_{13}$ and $v_{32}v_{31}v_{33}$. This triangulation is symmetric with respect to 180° rotation ρ about a line perpendicular to the square $v_{11}v_{12}v_{31}v_{32}$. It exchanges τ_1 and τ_3, and brings τ_2 to itself.

A *normal surface* is a properly embedded surface in W such that it is a union of normal discs, which are triangle discs and square discs in tetrahedra as in Figure 2. A properly embedded disc in a polyhedron is a *normal disk* if its edges are *normal arcs*, (i.e., they

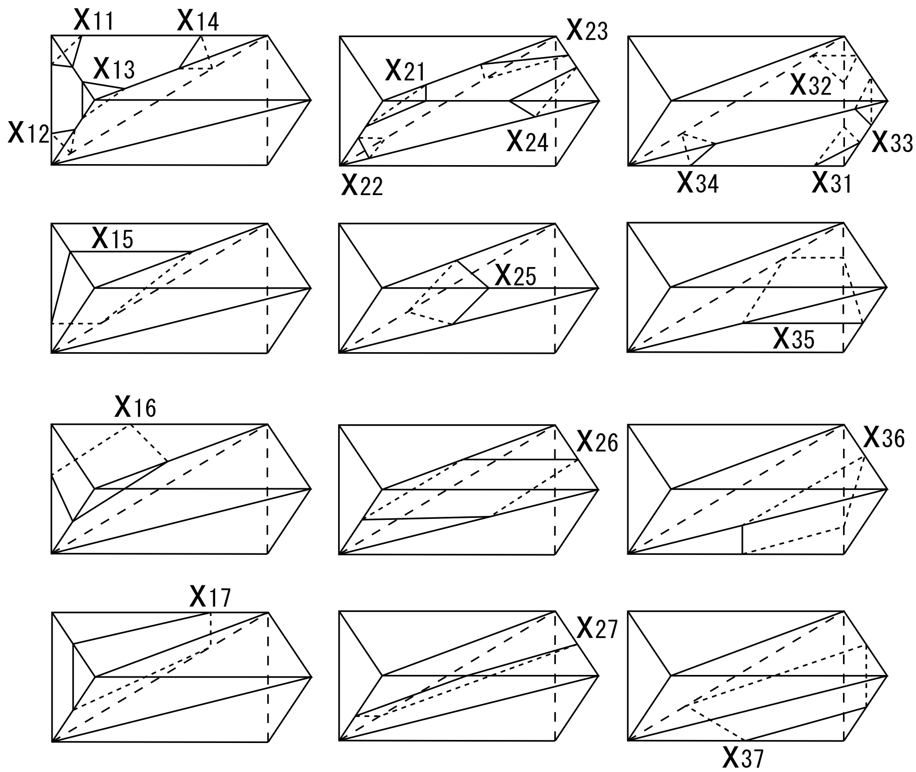

FIGURE 2. Normal disks in the triangulated solid torus

connect distinct edges of faces of the polyhedron), and if it intersects each edge of the polyhedron at most once. The tetrahedron τ_i contains 4 types of triangle normal disks $X_{i1}, X_{i2}, X_{i3}, X_{i4}$ and 3 types of square normal disks X_{i5}, X_{i6}, X_{i7}. The 180° rotation ρ exchanges the normal disks X_{1i} and X_{3i} for $1 \leq i \leq 7$. In the tetrahedron τ_2, X_{21} and X_{24} are exchanged by ρ, and X_{22} and X_{23} are also. Each of X_{25}, X_{26}, X_{27} is symmetric under ρ.

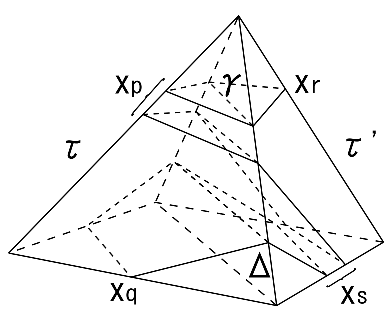

FIGURE 3. Matching equations

A set of normal disks forms a normal surfaces if it satisfies the matching equations. In general, a single matching equation arises for each type of normal arc γ in a 2-simplex Δ

which is an amalgamation of two faces of tetrahedra τ, τ'. (It is possible that $\tau = \tau'$.) Let X_p and X_q (resp. X_r, X_s) are the triangle type normal disk and the square type normal disk in τ (resp. τ') such that they have edges of type γ. We denote the number of the disks of types X_p, X_q, X_r, X_s by x_p, x_q, x_r, x_s. Then the *matching equation* on γ is $x_p + x_q = x_r + x_s$.

Considering all such normal arcs, we obtain simultaneous equations, called the matching equations. In our case, the coefficient matrix A is

$$\left[\begin{array}{ccccccc|ccccccc|ccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0
\end{array}\right].$$

Let $\mathbf{x} = {}^t(x_{11}\ x_{12}\ \cdots\ x_{17} \mid x_{21}\ x_{22}\ \cdots\ x_{27} \mid x_{31}\ x_{32}\ \cdots\ x_{37})$, the list of the number of normal disks. Then we have $A\mathbf{x} = \mathbf{0}$. (We should ignore the vertical lines between the 7th and the 8th columns and between the 14th and the 15th ones.) It is a routine work to see $\mathrm{rank}(M) = 9$, full rank, by elementary transformations of rows. Hence the solution space V of the matching equations in $\mathbb{R}^{21}$ is a $7 \times 3 - 9 = 12$ dimensional subspace.

Remark 2.1. In general, every matching equation is balanced, so that the sum of the entries of every row of a coefficient matrix is 0. This still holds after a sequence of elementary transformations of rows are applied, giving a method of checking calculation by hand.

One may expect that every column of A has 3 or 4 non-zero entries according to it corresponds to triangle type of normal disk or to square one. But it is not true when the same type of normal disk appears both sides of a matching equation. See the $(4, 1)$-lens space in section 5.

Lemma 2.2. *Let $\mathcal{T}$ be a triangulation of a 3-manifold M with t tetrahedra, and $\mathcal{A}$ the matching equations. Suppose that $\mathcal{T}$ has a 2-simplex Δ such that no pair of edges of Δ are amalgamated to each other and no edge of Δ is contained in the boundary ∂M. Let $\mathcal{A}'$ be the simultaneous equations obtained from $\mathcal{A}$ by deleting the three equations corresponding to the normal arcs in Δ. Then the solution space of $\mathcal{A}'$ in $\mathbb{R}^{7t}$ coincides with that of $\mathcal{A}$.*

Proof. For simplicity of notations, we only show that an admissible solution of $\mathcal{A}'$ in $\mathbb{Z}_+^{7t}$ is also an admissible slution of $\mathcal{A}$. Similar argument will do for the other solutions.

Let $\mathbf{p}'$ be a non-trivial admissible solution of $\mathcal{A}'$ such that each entry of $\mathbf{p}'$ is a non-negative integer. Then there is a surface F'_p corresponding to $\mathbf{p}'$ in the 3-dimensional complex M'

obtained by cutting M along Δ. (If we remove small open neighbourhoods of the vertices from M', then we obtain a 3-manifold, in which F'_p is properly embedded.)

Let e be an edge of Δ. There is a sequence of 2-simplices $\Delta = \Delta_1, \Delta_2, \cdots, \Delta_m$ containing the edge e and appearing in this order around e. For $1 \leq i \leq m$, let e_i be the copy of the edge e in Δ_i. Since e is not contained in ∂M, there is a tetrahedron τ_i between Δ_i and Δ_{i+1}, where $\Delta_{m+1} = \Delta_1$. Then matching equations on Δ_i guarantee that the number of points $F'_p \cap e_{i-1}$ is equal to that of $F'_p \cap e_i$ for $2 \leq i \leq m$. Note that $\Delta_i \neq \Delta$ for $i \neq 1$ because Δ has no pair of edges amalgamated.

Let f_1, f_1 be the copies of Δ in τ_1, τ_m. Then the above argument shows that $F'_p \cap \partial f_1$ and $F'_p \cap \partial f_2$ match. Hence the sets of normal arcs $F'_p \cap f_1$ and $F'_p \cap f_2$ also match. (Note that the set of normal arcs is uniquely determined by the number of the end points of them in the three edges since the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is nonsingular.) Thus $\mathbf{p}'$ satisfies the matching equations on Δ, and is an admissible solution of $\mathcal{A}$. □

The $(4,1)$-lens space can be decomposed into a single tetrahedron such that every 2-simplex has two edges amalgamated. However, the matching equation is not full-rank. In this sense, the above lemma is not keen.

Problem 2.3. (1) In general, is the coefficient matrix full rank if all the edges are contained in the boundary of the 3-manifold?

(2) Is the coefficient matrix not full rank if the 3-manifold is closed?

In this case of the solid torus, 12 vectors below together form a basis of V.

$\mathbf{p}_a = {}^t(0\ 0\ 0\ 0\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0\ 0\ 0\ 1)$
$\mathbf{p}_b = {}^t(0\ 0\ 0\ 0\ 0\ 1\ 0 \mid 1\ 0\ 0\ 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_c = {}^t(0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_d = {}^t(0\ 0\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 0\ 1\ 0\ 0\ 0)$
$\mathbf{p}_e = {}^t(0\ 0\ 1\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0\ 1\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0)$
$\mathbf{p}_f = {}^t(0\ 1\ 0\ 0\ 0\ 0\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 0\ 0)$
$\mathbf{p}_g = {}^t(1\ 0\ 0\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0\ 1\ 0)$
$\mathbf{p}_h = {}^t(0\ 0\ 1\ 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0)$
$\mathbf{p}_i = {}^t(0\ 1\ 0\ 0\ 0\ 0\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 1\ 0\ 0\ 0)$
$\mathbf{p}_j = {}^t(1\ 0\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_k = {}^t(0\ 0\ 0\ 0\ 0\ 0\ 1 \mid 0\ 1\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 1\ 0\ 0\ 0)$
$\mathbf{p}_l = {}^t(0\ 0\ 1\ 1\ 0\ 0\ 0 \mid 1\ 0\ 1\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0\ 0\ 1)$

In general, we call a basis as in the next lemma *simple*.

Notation 2.4. Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors of the same dimension n. We write $\mathbf{v} \geq \mathbf{w}$ if $v_i \geq w_i$ holds for all the i-th entries with $1 \leq i \leq n$. Moreover, $\mathbf{v} > \mathbf{w}$ means that $\mathbf{v} \geq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$, in which case $v_j > w_j$ for some $1 \leq j \leq n$.

Lemma 2.5. *In general, we can take a basis of the solution space of the matching equations so that it satisfy the two conditions below.*

(1) *The entries of each vector of the basis are coprime non-negative integers.*
(2) *The basis contains no pair of vectors* $\mathbf{p}$, $\mathbf{q}$ *with* $\mathbf{p} > \mathbf{q}$.

Proof. Since the coefficients of the matching equations are integers, we can take a basis so that the entries of each vector are coprime integers. We form a matrix with columns being the vectors of the basis. Elementary transformations of columns keep that the columns together form a basis of the solution space. We can make a column to be the vector $\tilde{\mathbf{x}} = {}^t(1 \ \cdots \ 1)$ with all its entries equal to 1 because it is always a solution of the matching equations. Then we can destroy all the negative entries by adding this column to the columns containing them. If the matrix contains two columns $\mathbf{p}$, $\mathbf{q}$ with $\mathbf{p} > \mathbf{q}$, then we subtract $\mathbf{q}$ from $\mathbf{p}$. ☐

Problem 2.6. Are there a triangulation of the solid torus and a simple basis of the solution space of the matching equations such that no vector of the basis represents a meridian disk?

Remark 2.7. For each tetrahedron τ_i, the vector $\mathbf{x}_i = {}^t(0 \ \cdots 0 \mid 1\,1\,1\,1 \ -1 \ -1 \ -1 \mid 0 \ \cdots 0)$ with all its entries of triangle types in τ_i equal to 1, those of square types in τ_i equal to -1 and the other ones equal to 0 is always a solution of the matching equations. Hence $(\tilde{\mathbf{x}} - \Sigma\epsilon_i\mathbf{x}_i)/2$ is also a solution, where $\epsilon_i = 1$ or -1 for each i. When $\epsilon_i = 1$ for all i, it is a union of all the disks and spheres which are links of the vertices of the triangulation.

A solution of the matching equations is called a *vertex solution* if its entries are coprime integers and if it is contained in a subspace W of dimension 1 such that W is intersection of the solution space of the matching equations and a subspace generated by fundamental vectors with a single entry equal to 1 and the other ones to 0.

If you can obtain all the vertex solutions by computer, then some of the vectors of the basis may be replaced with some of them, to make basis simpler.

Problem 2.8. Can we always take a basis of the solution space of the matching equations so that its vectors are all vertex solutions (possibly including non-admissible ones)?

In practice, it is better to draw figures of the normal surfaces of the vectors of basis. If there is a disconnected surface among them, then we can replace it with an adequate connected component of it. Moreover, connected components of the normal surface corresponding to the vectors in Remark 2.7 may substitute for some vectors of basis. Also "surrounding surfaces" of subcomplices of the triangulation may do. (Actually, the authors

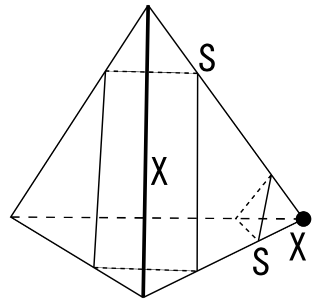

FIGURE 4. The surrounding surface of a subcomplex X

replaced (1 0 0 0 0 1 0 | 0 0 0 0 0 1 0 | 1 0 0 0 0 1 0) with $\mathbf{p}_e$ in this manner when they calculated the vectors of basis. The annulus $\mathbf{p}_e$ surrounds the loop edge $v_{21}v_{24}$.)

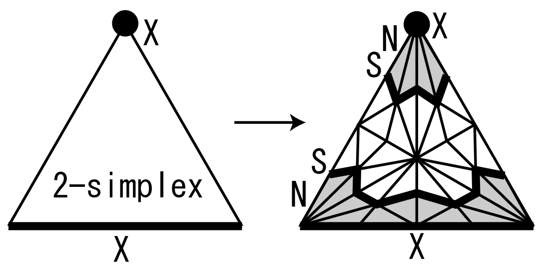

FIGURE 5. A schematic figure in dimension 2

In general, let X be a subcomplex of a triangulation $\mathcal{T}$ of a 3-manifold. We can form the surrounding surface of X as below. Let $\mathcal{T}'$ be the barycentric subdivision of $\mathcal{T}$, and $\mathcal{T}''$ that of $\mathcal{T}'$. We take a neighbourhood N of X so that N is the union of the tetrahedra of $\mathcal{T}''$ which are not disjoint from X. See Figure 5. Then the surrounding surface S of X is the union of the 2-simplices in N which are disjoint from X. If X is a vertex, then the surrounding surface of X is a vertex-linking sphere. The next lemma is very clear.

Lemma 2.9. *If a subcomplex X of $\mathcal{T}$ satisfies the two conditions below, then the surrounding surface of X forms a normal surface with respect to $\mathcal{T}$.*

(1) *For each 2-simplex Δ of $\mathcal{T}$, X contains Δ if X contains two edges of it.*
(2) *For each tetrahedron τ of $\mathcal{T}$, X contains τ if X contains the four faces of τ.*

A normal surface is called a fundamental surface if the corresponding vector $\mathbf{p}$ does not satisfy $\mathbf{p} = \mathbf{q} + \mathbf{r}$ for any pair of vectors $\mathbf{q}$, $\mathbf{r}$ corresponding to non-empty normal surfaces.

Lemma 2.10. *If a subcomplex X of $\mathcal{T}$ does not intersect any 2-simplex precisely in an edge e and a vertex disjoint from e, then every connected component of the surrounding surface S of X is a fundamental surface.*

Proof. If there is a 2-simplex Δ such that $S \cap \Delta$ contains a pair of parallel normal arcs of the same type, then X intersects Δ precisely in an edge e and a vertex v disjoint from e. Hence there is not such a pair of normal arcs. Then the next fact implies that every componenet of S is fundamental. □

Fact 2.11. *Let F be a connected normal surface. If F has at most one arc for every type of normal arc, then F is fundamental.*

Proof. The way of glueing the normal disks is unique, and hence F is not a Haken sum of other normal surfaces. □

We see later examples of fundamental surfaces which is a component of a surrounding surface and does not satisfy the condition of the above lemma.

The result in [6] implies that there is a fundamental surface which is not a component of a link of a subcomplex.

Problem 2.12. Is there a connected component of a link of a subcomplex as in Lemma 2.9 such that it is not a fundamental surface?

We prepare lemmas for calculating fundamental surfaces. They are very clear and will be used repeatedly in the calculation.

Lemma 2.13. *Let $\mathbf{p}$ be a vector corresponding to a fundamental surface, and $\mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_m$ vectors corresponding to normal surfaces. If $\mathbf{p} = z_1\mathbf{p}_1 + z_2\mathbf{p}_2 + \cdots + z_m\mathbf{p}_m$ for some non-negative integers $z_1, z_2, \cdots, z_m$, then $z_i = 1$ for some $1 \leq i \leq m$ and $z_j = 0$ for all $j \neq i$, $1 \leq j \leq m$.*

Lemma 2.14. *Let $\mathbf{p}$ be a vector corresponding to a fundamental surface F_p, and $\mathbf{q}$ a vector corresponding to a normal surface F_q. If $\mathbf{p} \geq \mathbf{q}$, then $\mathbf{p} = \mathbf{q}$.*

Proof. Clearly, $\mathbf{p} = (\mathbf{p} - \mathbf{q}) + \mathbf{q}$, and $\mathbf{p} - \mathbf{q}$ is a solution of the matching equations. If $\mathbf{p} > \mathbf{q}$, then $\mathbf{p} - \mathbf{q}$ is larger than $\mathbf{0}$ and represents a normal surface F. Hence F_p is a Haken sum of two normal surfaces F_q and F, contradicting that $\mathbf{p}$ represents a fundamental surface. □

Now we begin to calculate fundamental surfaces in the triangulated solid torus. Let $\mathbf{p}$ be a solution in V. Then

$$\begin{aligned}\mathbf{p} &= a\mathbf{p}_a + b\mathbf{p}_b + \cdots + l\mathbf{p}_l \\ &= {}^t(g+j \quad f+i \quad e+h+l \quad d+j+l \quad c \quad b \quad a+k \quad | \\ &\qquad b+h+l \quad f+i+k \quad c+j+l \quad g+h+k \quad d \quad e \quad a \quad | \\ &\qquad\qquad b+i \quad c+j \quad e+h+k \quad d+i+k \quad f \quad g \quad a+l),\end{aligned}$$

for some set of real numbers $a, b, \cdots, l$. If $\mathbf{p}$ represents a normal surface, then all the entries are non-negative integers. We can see a, b, c, d, e, f, g are integers from the 5th, 6th, 12th, 13th, 14th, 19th and 20th entries. Then i, j, k, l are integers since the 1st, 2nd, 7th, 21st entries are integers. Hence h is also an integer from the 3rd entry. Thus all the arbitarary constants $a, b, \cdots, l$ are integers.

Problem 2.15. Are there a triangulation of a compact 3-manifold and a simple basis of the solution space of the matching equations such that there is a normal surface which is not equal to a linear combination of the vectors of the basis with integer coefficients?

A normal surface does not have two normal disks of distinct square types in a tetrahedron. (If it did, then it would have self-intersection.) Hence one of the conditions $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$ below holds for the tetrahedron τ_1: $\langle 1 \rangle$ the 6th and 7th entries are zero, $\langle 2 \rangle$ the 5th and 7th entries are zero, and $\langle 3 \rangle$ the 5th and 6th entries are zero. For τ_2, [1] the 13th and 14th entries are zero, [2] the 12th and 14th entries are zero, or [3] the 12th and 13th entries are zero. Moreover, (1) the 20th and 21st entries are zero, (2) the 19th and 21st entries are zero, or (3) the 19th and 20th entries are zero for τ_3. Thus we have $3 \times 3 \times 3 = 27$ cases. We call the case $\langle$i$\rangle$ and [j] and (k) the case i-j-k for short. Since the cases i-j-k and k-j-i are symmetric under the 180° rotation, it is enough to consider the only 18 cases i-j-k with $i \leq k$. We will show only 5 cases 1-1-1, 2-2-2, 1-1-2, 1-3-3, 3-3-3 because similar calculation will do for the other cases.

In the case 1-1-1, $b, a+k, e, a, g, a+l = 0$, and hence $k, l = 0$. Substituting 0 for a, b, e, g, k, l in the general solution, we have
$\mathbf{p} = c\mathbf{p}_c + d\mathbf{p}_d + f\mathbf{p}_f + h\mathbf{p}_h + i\mathbf{p}_i + j\mathbf{p}_j$
$= {}^t(j \;\; f+i \;\; h \;\; d+j \;\; c \;\; 0 \;\; 0 \mid h \;\; f+i \;\; c+j \;\; h \;\; d \;\; 0 \;\; 0 \mid i \;\; c+j \;\; h \;\; d+i \;\; f \;\; 0 \;\; 0)$.
Since all the entries are non-negative, $c, d, f, h, i, j \geq 0$ from the 5th, 12th, 19th, 3rd, 15th and 1st entries. If $\mathbf{p}$ is fundamental, then precisely one of c, d, f, h, i, j is equal to 1 and the others are 0 by Lemma 2.13. Thus, in the case 1-1-1, $\mathbf{p} = \mathbf{p}_c, \mathbf{p}_d, \mathbf{p}_f, \mathbf{p}_h, \mathbf{p}_i$ or $\mathbf{p}_j$.

In the case 2-2-2, $c, a+k, d, a, f, a+l = 0$, and hence $k, l = 0$. Substituting 0 for a, c, d, f, k, l in $\mathbf{p}$, we have
$\mathbf{p} = {}^t(g+j \;\; i \;\; e+h \;\; j \;\; 0 \;\; b \;\; 0 \mid b+h \;\; i \;\; j \;\; g+h \;\; 0 \;\; e \;\; 0 \mid b+i \;\; j \;\; e+h \;\; i \;\; 0 \;\; g \;\; 0)$.
Since all the entries are non-negative, $b, e, g, i, j \geq 0$ from the 6th, 13th, 20th, 2nd and 4th entries. Suppose $\mathbf{p}$ is fundamental. If $h \geq 0$, then $\mathbf{p} = b\mathbf{p}_b + e\mathbf{p}_e + g\mathbf{p}_g + h\mathbf{p}_h + i\mathbf{p}_i + j\mathbf{p}_j$ with $b, e, g, h, i, j \geq 0$. Hence Lemma 2.13 implies that $\mathbf{p} = \mathbf{p}_b, \mathbf{p}_e, \mathbf{p}_g, \mathbf{p}_h, \mathbf{p}_i$ or $\mathbf{p}_j$. We can assume $h < 0$. If $\mathbf{p} \geq \mathbf{p}_b$ then $\mathbf{p} = \mathbf{p}_b$ by Lemma 2.14. Hence we can assume that $\mathbf{p} \not\geq \mathbf{p}_b$. The non-zero entries of $\mathbf{p}_b$ are the 6th, 8th and 15th entries, and they are 1. Therefore, among the corresponding entries of $\mathbf{p}$, the smallest one $b+h$ is equal to 0. Similarly, comparing $\mathbf{p}$ with $\mathbf{p}_e$ and $\mathbf{p}_g$, Lemma 2.14 implies that $\mathbf{p} = \mathbf{p}_e, \mathbf{p}_g$ or $e+h = 0$ and $g+h = 0$. We consider the case of $b, e, g = -h$, where
$\mathbf{p} = {}^t(j-h \;\; i \;\; 0 \;\; j \;\; 0 \;\; -h \;\; 0 \mid 0 \;\; i \;\; j \;\; 0 \;\; 0 \;\; -h \;\; 0 \mid i-h \;\; j \;\; 0 \;\; i \;\; 0 \;\; -h \;\; 0)$.
Recall that $i, j \geq 0$ and $-h > 0$. When $i = 0$, $j = 0$ and $-h = 1$, $\mathbf{p}$ is equal to

$\mathbf{p}_q = {}^t(1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \mid\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0)$.
Because $\mathbf{p} \geq \mathbf{p}_q$, Lemma 2.14 implies that $\mathbf{p} = \mathbf{p}_q$. Thus, in the case 2-2-2, $\mathbf{p} = \mathbf{p}_b, \mathbf{p}_e, \mathbf{p}_g, \mathbf{p}_h, \mathbf{p}_i, \mathbf{p}_j$ or $\mathbf{p}_q$.

In the case 1-1-2, $b, a+k, e, a, f, a+l = 0$, and hence $k, l = 0$. Substituting 0 for a, b, e, f, k, l in $\mathbf{p}$, we have
$\mathbf{p} = {}^t(g+j\ \ i\ \ h\ \ d+j\ \ c\ \ 0\ \ 0\ \mid\ h\ \ i\ \ c+j\ \ g+h\ \ d\ \ 0\ \ 0\ \mid\ i\ \ c+j\ \ h\ \ d+i\ \ 0\ \ g\ \ 0)$.
Since all the entries are non-negative, $c, d, g, h, i \geq 0$ from the 5th, 12th, 20th, 3rd and 2nd entries. Suppose $\mathbf{p}$ is fundamental. Comparing $\mathbf{p}$ with $\mathbf{p}_h$, we can see that $\mathbf{p} = \mathbf{p}_h$ or $h = 0$ by Lemma 2.14 since among the 3rd entry h, 8th entry h, the 11th entry $g+h$ and the 17th entry h the smallest is h. Similarly, comparing $\mathbf{p}$ with $\mathbf{p}_i$, Lemma 2.14 implies that $\mathbf{p} = \mathbf{p}_i$ or $i = 0$. We consider the case of $h = 0$ and $i = 0$, where
$\mathbf{p} = {}^t(g+j\ \ 0\ \ 0\ \ d+j\ \ c\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ c+j\ \ g\ \ d\ \ 0\ \ 0\ \mid\ 0\ \ c+j\ \ 0\ \ d\ \ 0\ \ g\ \ 0)$.
If $j \geq 0$, then $\mathbf{p} = c\mathbf{p}_c + d\mathbf{p}_d + g\mathbf{p}_g + j\mathbf{p}_j$ with $c, d, g, j \geq 0$. Hence $\mathbf{p} = \mathbf{p}_c, \mathbf{p}_d, \mathbf{p}_g$ or $\mathbf{p}_j$ by Lemma 2.13. Thus we can assume $j < 0$. Comparing $\mathbf{p}$ with $\mathbf{p}_c$, we can see that $\mathbf{p} = \mathbf{p}_c$ or $\mathbf{p} \not\geq \mathbf{p}_c$ by Lemma 2.14. In the latter case we have $c + j = 0$ since $c > c + j$. Similarly, comparing $\mathbf{p}$ with $\mathbf{p}_d$ and $\mathbf{p}_g$, we have $\mathbf{p} = \mathbf{p}_d, \mathbf{p}_g$ or $d + j = 0$ and $g + j = 0$. In the last case, replacing c, d, g with $-j$ in $\mathbf{p}$, we have
$\mathbf{p} = {}^t(0\ \ 0\ \ 0\ \ 0\ \ -j\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ -j\ \ -j\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ -j\ \ 0\ \ -j\ \ 0)$.
Only when $-j = 1$, this vector is fundamental and $\mathbf{p}$ is equal to
$\mathbf{p}_m = {}^t(0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0)$.
Thus, in the case 1-1-2, $\mathbf{p} = \mathbf{p}_c, \mathbf{p}_d, \mathbf{p}_g, \mathbf{p}_h, \mathbf{p}_i, \mathbf{p}_j$ or $\mathbf{p}_m$.

In the case 1-3-3, $b, a+k, d, e, f, g = 0$. Replacing b, d, e, f, g with 0, and k with $-a$ in the general solution, we have
$\mathbf{p} = {}^t(j\ \ i\ \ h{+}l\ \ j{+}l\ \ c\ \ 0\ \ 0\ \mid\ h{+}l\ \ i{-}a\ \ c{+}j{+}l\ \ h{-}a\ \ 0\ \ 0\ \ a\ \mid\ i\ \ c{+}j\ \ h{-}a\ \ i{-}a\ \ 0\ \ 0\ \ a{+}l)$.
Since all the entries are non-negative, $a, c, i, j \geq 0$ from the 14th, 5th, 2nd and 1st entries. In addition, $l \geq -a$ from the 21st entry. Suppose that $\mathbf{p}$ is fundamental. If $\mathbf{p} \geq \mathbf{p}_i$, then $\mathbf{p} = \mathbf{p}_i$ by Lemma 2.14. We can assume $\mathbf{p} \not\geq \mathbf{p}_i$. Since non-zero entries of $\mathbf{p}_i$ are the 2nd, 9th, 15th and 18th entries and they are 1, among the corresponding entries of $\mathbf{p}$ the smallest one $i - a$ is equal to 0. Comparing $\mathbf{p}$ with $\mathbf{p}_h$, a similar argument shows that $\mathbf{p} = \mathbf{p}_h$ or $h = a$. (Note that $h + l \geq h - a$ because $l \geq -a$.) If $l > 0$, then $\mathbf{p} \geq \mathbf{p}_l$ since $h = a \geq 0$. Then $\mathbf{p} = \mathbf{p}_l$ by Lemma 2.14. We can assume $l \leq 0$. Hence $j + l \leq j$. Comparing $\mathbf{p}$ with $\mathbf{p}_j$, a similar argument as above shows that $\mathbf{p} = \mathbf{p}_j$ or $j = -l$. Substituting a, a, $-l$ for i, h, j respectively in $\mathbf{p}$, we obtain
$\mathbf{p} = {}^t(-l\ \ a\ \ a+l\ \ 0\ \ c\ \ 0\ \ 0\ \mid\ a+l\ \ 0\ \ c\ \ 0\ \ 0\ \ 0\ \ a\ \mid\ a\ \ c-l\ \ 0\ \ 0\ \ 0\ \ 0\ \ a+l)$.
Comparing $\mathbf{p}$ with $\mathbf{p}_c$, we can see that $\mathbf{p} = \mathbf{p}_c$ or $c = 0$. In the latter case, we have
$\mathbf{p} = {}^t(-l\ \ a\ \ a+l\ \ 0\ \ 0\ \ 0\ \ 0\ \mid\ a+l\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ a\ \mid\ a\ \ -l\ \ 0\ \ 0\ \ 0\ \ 0\ \ a+l)$.
From the 3rd entry $a \geq -l (\geq 0)$. When $a = 1$ and $l = 0$, we have $\mathbf{p} = \mathbf{p}_t$ as below.

$\mathbf{p}_t = {}^t(0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \mid\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \mid\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1)$.
When $a = 1$ and $l = -1$, we have $\mathbf{p} = \mathbf{p}_r$ as below.
$\mathbf{p}_r = {}^t(1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \mid\ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)$.
Since $\mathbf{p} = (a + l)\mathbf{p}_t + (-l)\mathbf{p}_r$ with $a + l, -l \geq 0$, Lemma 2.13 implies that $\mathbf{p} = \mathbf{p}_t$ or $\mathbf{p}_r$. Thus, in the case 1-3-3, $\mathbf{p} = \mathbf{p}_c, \mathbf{p}_h, \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_l, \mathbf{p}_r$ or $\mathbf{p}_t$.

In the case 3-3-3, replacing b, c, d, e, f, g with 0 in $\mathbf{p}$, we have
$\mathbf{p} = {}^t(j\ \ i\ \ h{+}l\ \ j{+}l\ \ 0\ \ 0\ \ a{+}k\ \mid\ h{+}l\ \ i{+}k\ \ j{+}l\ \ h{+}k\ \ 0\ \ 0\ \ a\ \mid\ i\ \ j\ \ h{+}k\ \ i{+}k\ \ 0\ \ 0\ \ a{+}l)$.
Because all the entries are non-negative, $a, i, j \geq 0$ from the 14th, 2nd and 1st entries. We consider the case of $k \geq l$ only, omitting the case of $k < l$, which is symmetric to the case $k > l$ by the 180° rotation. (Note that 3-3-3 is a symmetric condition under the 180° rotation.) Comparing $\mathbf{p}$ with $\mathbf{p}_h$, Lemma 2.14 implies that $\mathbf{p} = \mathbf{p}_h$ or $\mathbf{p} \not\geq \mathbf{p}_h$. In the latter case we have $h + l = 0$ because $k \geq l$. Replacing h with $-l$, we have
$\mathbf{p} = {}^t(j\ \ i\ \ 0\ \ j+l\ \ 0\ \ 0\ \ a+k\ \mid\ 0\ \ i+k\ \ j+l\ \ k-l\ \ 0\ \ 0\ \ a\ \mid\ i\ \ j\ \ k-l\ \ i+k\ \ 0\ \ 0\ \ a+l)$.

We consider first the case of $l > 0$. Recall that $a, i, j \geq 0$ and $k \geq l > 0$. If we set $a = 0, i = 0, j = 0, k = 1$ and $l = 1$ in $\mathbf{p}$, then we obtain the vector
$\mathbf{p}_s = {}^t(0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \mid\ 0\ \ 1\ \ 1 0\ \ 0\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1)$.
Hence $\mathbf{p} \geq \mathbf{p}_s$, and Lemma 2.14 implies that $\mathbf{p} = \mathbf{p}_s$.

If $l = 0$, then $\mathbf{p} = \mathbf{p}_a, \mathbf{p}_h, \mathbf{p}_i, \mathbf{p}_j$ or $\mathbf{p}_k$ by Lemma 2.13. Hence we can assume that $l < 0$. Comparing $\mathbf{p}$ with $\mathbf{p}_a$, Lemma 2.14 implies that $\mathbf{p} = \mathbf{p}_a$ or $\mathbf{p} \not\geq \mathbf{p}_a$. In the latter case we have $a + l = 0$ because $a, a + k \geq a + l$. Similarly $\mathbf{p} = \mathbf{p}_j$ or $j + l = 0$ by Lemma 2.14. Substituting $-l$ for a, j in $\mathbf{p}$ we have
$\mathbf{p} = {}^t(-l\ \ i\ \ 0\ \ 0\ \ 0\ \ 0\ \ k-l\ \mid\ 0\ \ i+k\ \ 0\ \ k-l\ \ 0\ \ 0\ \ -l\ \mid\ i\ \ -l\ \ k-l\ \ i+k\ \ 0\ \ 0\ \ 0)$.
If $k \geq 0$, then $i, k, -l \geq 0$, and Lemma 2.13 implies that $\mathbf{p} = \mathbf{p}_i, \mathbf{p}_k$ or
$\mathbf{p}_u = {}^t(1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \mid\ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \mid\ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0)$.
We can assume that $k < 0$. Comparing $\mathbf{p}$ with $\mathbf{p}_i$ and $\mathbf{p}_u$, either $\mathbf{p} = \mathbf{p}_i$, $\mathbf{p}_u$ or $i + k = 0$ and $k - l = 0$ by Lemma 2.14. In the last case, replacing $i, -l$ with $-k$, we have
$\mathbf{p} = {}^t(-k\ \ -k\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -k\ \mid\ -k\ \ -k\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)$.
Only when $-k = 1$, this vector is fundamental and $\mathbf{p} = \mathbf{p}_r$.

In general, see the more systematic method in section 4 for more complicated cases.

Thus we can obtain candidates of fundamental surfaces $\mathbf{p}_a, \mathbf{p}_b, \cdots, \mathbf{p}_m,\ \mathbf{p}_q, \mathbf{p}_r, \cdots, \mathbf{p}_u$. We must add one more candidate
$\mathbf{p}_n = {}^t(0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \mid\ 1\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \mid\ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0)$
which is symmetric to $\mathbf{p}_m$ by the 180° rotation. The set of these candidate vectors contains all the fundamental surfaces.

We should confirm that these vectors are really fundamental. By Fact 2.11 $\mathbf{p}_a, \mathbf{p}_b, \cdots, \mathbf{p}_l$ are fundamental. However, the other vectors do not satisfy the condition of Fact 2.11.

In general, let $\sigma(\mathbf{v})$, the size of a vector $\mathbf{v}$, be the sum of the entries of $\mathbf{v}$. All the cancidates are of size larger than or equal to 3. If a vector is not fundamental, then it is a sum of fundamental ones, and hence is of size larger than or equal to 6. Thus $\mathbf{p}_m, \mathbf{p}_n, \mathbf{p}_q, \mathbf{p}_r$ are fundamental.

Suppose, for a contradiction, $\mathbf{p}_t$ is not fundamental. Since $\sigma(\mathbf{p}_t) = 6$, it is a sum of two vectors of size 3. The 8th entry of $\mathbf{p}_t$ is 1. Among the candidates, there is only one vector $\mathbf{p}_b$ of size 3 with non-zero 8th entry. Hence $\mathbf{p}_t = \mathbf{p}_b + \mathbf{q}$ for some normal surface $\mathbf{q}$. This contradicts the fact $\mathbf{p}_t \not> \mathbf{p}_b$. Thus $\mathbf{p}_t$ is fundamental.

A similar argument shows that $\mathbf{p}_s$ is fundamental. $\mathbf{p}_u$ is fundamental because it is symmetric to $\mathbf{p}_t$ by the 180° rotation.

Theorem 2.16. *The triangulated solid torus has precisely* 19 *fundamental surfaces below.*

$\mathbf{p}_a = {}^t(0\ 0\ 0\ 0\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0\ 0\ 0\ 1)$
$\mathbf{p}_b = {}^t(0\ 0\ 0\ 0\ 0\ 1\ 0 \mid 1\ 0\ 0\ 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_c = {}^t(0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_d = {}^t(0\ 0\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 0\ 1\ 0\ 0\ 0)$
$\mathbf{p}_e = {}^t(0\ 0\ 1\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0\ 1\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0)$
$\mathbf{p}_f = {}^t(0\ 1\ 0\ 0\ 0\ 0\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 0\ 0)$
$\mathbf{p}_g = {}^t(1\ 0\ 0\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0\ 1\ 0)$
$\mathbf{p}_h = {}^t(0\ 0\ 1\ 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0)$
$\mathbf{p}_i = {}^t(0\ 1\ 0\ 0\ 0\ 0\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 1\ 0\ 0\ 0)$
$\mathbf{p}_j = {}^t(1\ 0\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_k = {}^t(0\ 0\ 0\ 0\ 0\ 0\ 1 \mid 0\ 1\ 0\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 1\ 0\ 0\ 0)$
$\mathbf{p}_l = {}^t(0\ 0\ 1\ 1\ 0\ 0\ 0 \mid 1\ 0\ 1\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0\ 0\ 1)$
$\mathbf{p}_m = {}^t(0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 0\ 1\ 1\ 0\ 0 \mid 0\ 0\ 0\ 1\ 0\ 1\ 0)$
$\mathbf{p}_n = {}^t(0\ 0\ 0\ 1\ 0\ 1\ 0 \mid 1\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 0\ 0)$
$\mathbf{p}_q = {}^t(1\ 0\ 0\ 0\ 0\ 1\ 0 \mid 0\ 0\ 0\ 0\ 0\ 1\ 0 \mid 1\ 0\ 0\ 0\ 0\ 1\ 0)$
$\mathbf{p}_r = {}^t(1\ 1\ 0\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0\ 0\ 1 \mid 1\ 1\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_s = {}^t(0\ 0\ 0\ 1\ 0\ 0\ 1 \mid 0\ 1\ 1\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 1\ 0\ 0\ 1)$
$\mathbf{p}_t = {}^t(0\ 1\ 1\ 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 0\ 0\ 0\ 1 \mid 1\ 0\ 0\ 0\ 0\ 0\ 1)$
$\mathbf{p}_u = {}^t(1\ 0\ 0\ 0\ 0\ 0\ 1 \mid 0\ 0\ 0\ 1\ 0\ 0\ 1 \mid 0\ 1\ 1\ 0\ 0\ 0\ 0)$

Remark 2.17. (1) $\mathbf{p}_a$ is a Möbius band over which the solid torus is a twisted I-bundle.

(2) $\mathbf{p}_b, \mathbf{p}_d, \mathbf{p}_g, \mathbf{p}_k, \mathbf{p}_l, \mathbf{p}_r$ are meridian disks. $\mathbf{p}_b, \mathbf{p}_d, \mathbf{p}_g$ are parallel to the 2-simplices $v_{22}v_{23}v_{24}$, $v_{11}v_{12}v_{13}$, $v_{21}v_{22}v_{23}$ respectively. Their doubles $2\mathbf{p}_b, 2\mathbf{p}_d, 2\mathbf{p}_g$ surround these 2-simplex.

(3) $\mathbf{p}_c, \mathbf{p}_e, \mathbf{p}_f$ are annuli surrounding the edges $v_{11}v_{14}$, $v_{21}v_{24}$, $v_{31}v_{34}$ respectively.

(4) $\mathbf{p}_h, \mathbf{p}_i, \mathbf{p}_j$ are disks linking the vertices v_{13}, v_{12}, v_{11} respectively.

(5) $\mathbf{p}_m, \mathbf{p}_n, \mathbf{p}_q, \mathbf{p}_s, \mathbf{p}_t, \mathbf{p}_u$ are disks surrounding the edges $v_{12}v_{13}$, $v_{11}v_{13}$, $v_{12}v_{14}$, $v_{11}v_{12}$, $v_{22}v_{24}$, $v_{13}v_{14}$.

We can see by computer that these fundamental surfaces correspond to vertex solutions of the matching equations. (We confirmed this by the program for *Mathematica* shown in Appendix. Because it is difficult for the authors to install Linux, where *Regina* runs.) In fact, all the fundamental surfaces which we find in this paper correspond to vertex solutions.

Problem 2.18. Give a concrete example of a fundamental surface which does not correspond to any vertex solution of the matching equations.

Exercise 2.19. We can obtain another triangulation of the solid torus from the above triangulation by amalgamating the squares $v_{11}v_{12}v_{31}v_{14}$ and $v_{21}v_{22}v_{31}v_{24}$. Calculate fundamental surfaces for this triangulation.

3. 3-dimensional torus

We divide the 3-torus $T^3 = S^1 \times S^1 \times S^1$ into 6 tetrahedra, and calculate all the fundamental surfaces. In this case, the calculation is very easy and beautiful. Hence we can show the whole of it.

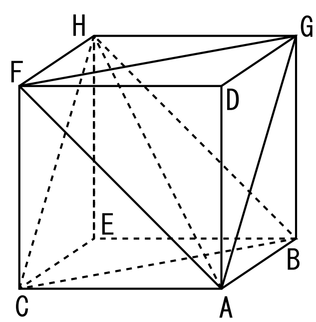

Figure 6. A triangulation of the 3-torus

We obtain the 3-torus from the cube in Figure 6 by glueing the pair of rectangles $ABEC$ and $DGHF$, the pair $ABGD$ and $CEHF$ and the pair $ACFD$ and $BEHG$. The cube is decomposed into 6 tetrahedra $\tau_1 = ADFG$, $\tau_2 = AFGH$, $\tau_3 = ABGH$, $\tau_4 = ACFH$, $\tau_5 = ABCH$ and $\tau_6 = BCEH$, gving a triangulation of T^3. This triangulation is symmetric under the reflection in the plane $ADHE$ and under the 180° rotation about the line MN, where M, N are the middle points of the edges BG, CF respectively. We number the types of normal disks in tetrahedra τ_1, τ_2, τ_3 as in Figure 7. Then the types of normal diks in τ_5 and τ_6 are numbered so that the 180° rotation brings X_{ij} in τ_1, τ_2 to $X_{(7-i)j}$ in τ_6, τ_5. Similarly, the types of normal disks in τ_4 are numbered so that the reflection brings X_{3j} in τ_3 to X_{4j} in τ_4.

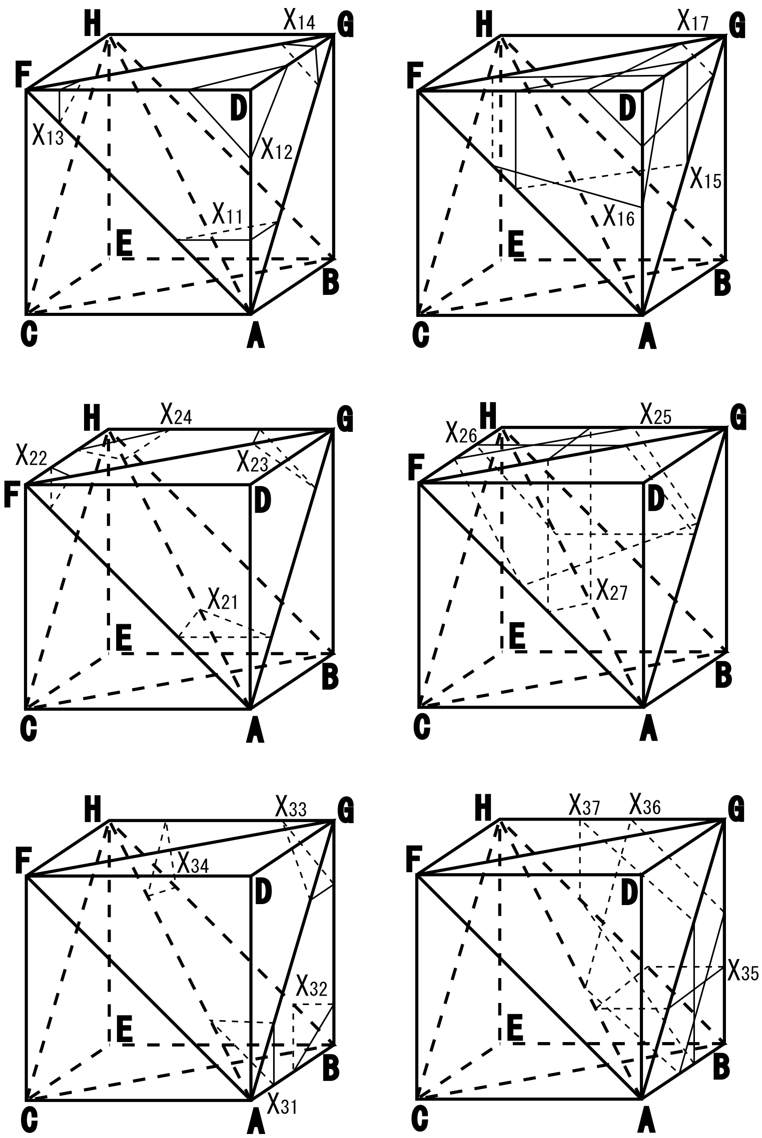

FIGURE 7. Number the types of normal disks

We can solve the mathcing equations in $\mathbb{R}^{42}$. It is a routine work to see that the 13 vectors below form a basis of the solution space.

$\mathbf{p}_a = {}^t(1\,0\,0\,0\,0\,0\,0 \mid 1\,0\,0\,0\,0\,0\,0 \mid 0\,0\,0\,0\,1\,0\,0 \mid 0\,0\,0\,0\,1\,0\,0 \mid 1\,0\,0\,0\,0\,0\,0 \mid 1\,0\,0\,0\,0\,0\,0)$

$\mathbf{p}_b = {}^t(0\,1\,0\,0\,0\,0\,0 \mid 0\,0\,0\,1\,0\,0\,0 \mid 0\,0\,0\,0\,0\,1\,0 \mid 0\,0\,0\,0\,0\,1\,0 \mid 0\,0\,0\,1\,0\,0\,0 \mid 0\,1\,0\,0\,0\,0\,0)$

$\mathbf{p}_c = {}^t(0\,0\,1\,0\,0\,0\,0 \mid 0\,0\,0\,0\,0\,0\,1 \mid 0\,0\,0\,1\,0\,0\,0 \mid 1\,0\,0\,0\,0\,0\,0 \mid 0\,0\,0\,0\,0\,1\,0 \mid 0\,0\,0\,1\,0\,0\,0)$

$\mathbf{p}_d = {}^t(0\,0\,0\,1\,0\,0\,0 \mid 0\,0\,0\,0\,0\,1\,0 \mid 1\,0\,0\,0\,0\,0\,0 \mid 0\,0\,0\,1\,0\,0\,0 \mid 0\,0\,0\,0\,0\,0\,1 \mid 0\,0\,1\,0\,0\,0\,0)$

$\mathbf{p}_e = {}^t(0\,0\,0\,0\,0\,1\,0 \mid 0\,1\,0\,0\,0\,0\,0 \mid 0\,1\,0\,0\,0\,0\,0 \mid 0\,0\,1\,0\,0\,0\,0 \mid 0\,0\,1\,0\,0\,0\,0 \mid 0\,0\,0\,0\,0\,0\,1)$

$\mathbf{p}_f = {}^t(0\,0\,0\,0\,0\,0\,1 \mid 0\,0\,1\,0\,0\,0\,0 \mid 0\,0\,1\,0\,0\,0\,0 \mid 0\,1\,0\,0\,0\,0\,0 \mid 0\,1\,0\,0\,0\,0\,0 \mid 0\,0\,0\,0\,0\,1\,0)$

$\mathbf{p}_g = {}^t(1\,1\,0\,0\,0\,0\,0 \mid 0\,0\,0\,0\,1\,0\,0 \mid 0\,1\,1\,0\,0\,0\,0 \mid 0\,1\,1\,0\,0\,0\,0 \mid 0\,0\,0\,0\,1\,0\,0 \mid 1\,1\,0\,0\,0\,0\,0)$

$\mathbf{p}_h = {}^t(0\,0\,1\,1\,0\,0\,0 \mid 0\,1\,1\,0\,0\,0\,0 \mid 0\,0\,0\,0\,0\,0\,1 \mid 0\,0\,0\,0\,0\,0\,1 \mid 0\,1\,1\,0\,0\,0\,0 \mid 0\,0\,1\,1\,0\,0\,0)$

$\mathbf{p}_i = {}^t(0\,0\,0\,0\,1\,0\,0 \mid 1\,0\,0\,1\,0\,0\,0 \mid 1\,0\,0\,1\,0\,0\,0 \mid 1\,0\,0\,1\,0\,0\,0 \mid 1\,0\,0\,1\,0\,0\,0 \mid 0\,0\,0\,0\,1\,0\,0)$
$\mathbf{p}_j = {}^t(1\,1\,1\,1\,0\,0\,0 \mid 1\,1\,1\,1\,0\,0\,0 \mid 1\,1\,1\,1\,0\,0\,0 \mid 0\,0\,0\,0\,1\,1\,1 \mid 0\,0\,0\,0\,1\,1\,1 \mid 0\,0\,0\,0\,1\,1\,1)$
$\mathbf{p}_k = {}^t(1\,1\,1\,1\,0\,0\,0 \mid 0\,0\,0\,0\,1\,1\,1 \mid 0\,0\,0\,0\,1\,1\,1 \mid 1\,1\,1\,1\,0\,0\,0 \mid 0\,0\,0\,0\,1\,1\,1 \mid 1\,1\,1\,1\,0\,0\,0)$
$\mathbf{p}_l = {}^t(1\,1\,1\,1\,0\,0\,0 \mid 0\,0\,0\,0\,1\,1\,1 \mid 0\,0\,0\,0\,1\,1\,1 \mid 0\,0\,0\,0\,1\,1\,1 \mid 1\,1\,1\,1\,0\,0\,0 \mid 1\,1\,1\,1\,0\,0\,0)$
$\mathbf{p}_m = {}^t(1\,1\,1\,1\,0\,0\,0 \mid 1\,1\,1\,1\,0\,0\,0 \mid 1\,1\,1\,1\,0\,0\,0 \mid 1\,1\,1\,1\,0\,0\,0 \mid 1\,1\,1\,1\,0\,0\,0 \mid 1\,1\,1\,1\,0\,0\,0)$

Remark 3.1. In the notation in Remark 2.7, $\mathbf{p}_a+\mathbf{p}_b+\mathbf{p}_h = (\tilde{\mathbf{x}}+\mathbf{x}_1+\mathbf{x}_2-\mathbf{x}_3-\mathbf{x}_4+\mathbf{x}_5+\mathbf{x}_6)/2$, $\mathbf{p}_c+\mathbf{p}_d+\mathbf{p}_g = (\tilde{\mathbf{x}}+\mathbf{x}_1-\mathbf{x}_2+\mathbf{x}_3+\mathbf{x}_4-\mathbf{x}_5+\mathbf{x}_6)/2$, $\mathbf{p}_e+\mathbf{p}_f+\mathbf{p}_i = (\tilde{\mathbf{x}}-\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3+\mathbf{x}_4+\mathbf{x}_5-\mathbf{x}_6)/2$, $\mathbf{p}_j = (\tilde{\mathbf{x}}+\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3-\mathbf{x}_4-\mathbf{x}_5-\mathbf{x}_6)/2$, $\mathbf{p}_k = (\tilde{\mathbf{x}}+\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3+\mathbf{x}_4-\mathbf{x}_5+\mathbf{x}_6)/2$, $\mathbf{p}_l = (\tilde{\mathbf{x}}+\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3-\mathbf{x}_4+\mathbf{x}_5+\mathbf{x}_6)/2$, $\mathbf{p}_m = (\tilde{\mathbf{x}}+\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3+\mathbf{x}_4+\mathbf{x}_5+\mathbf{x}_6)/2$.

Lemma 3.2. *The general solution $\mathbf{p} = a\mathbf{p}_a + b\mathbf{p}_b + \cdots m\mathbf{p}_m$ represents a normal surface if and only if it satisfies the following 4 conditions.*

1. *$a, b, \cdots, m$ are all integers.*
2. *$j = 0$, $k = 0$ and $l = 0$.*
3. *At least two of a, b, h, at least two of c, d, g and at least two of e, f, i are 0.*
4. (a) *($g = 0$ or $h = 0$ or $i = 0$) and $m \geq 0$, or*
   (b) *$g > 0$ and $h > 0$ and $i > 0$ and $m \geq max(-g, -h, -i)$.*

Proof. The general solution is of the form $\mathbf{p} = a\mathbf{p}_a + b\mathbf{p}_b + \cdots m\mathbf{p}_m$, where $a, b, \cdots, m$ are arbitrary constants. If this vector represents a normal surface, then all the entries are non-negative integers. From the 5th, 6th and 7th entries e, f, i are integers. Then j is an integer from the 42nd entry. Hence from the 35th entry $k + d$ is an integer, which and the 13th entry imply l is an integer. In this way, we can see that $a, b, \cdots, m$ are all integers. Thus (1) follows.

A normal surface does not have two types of square normal disks in any tetrahedron. Hence at least two of e, f, i are equal to 0 from the 5th, 6th and 7th entries. Then at least two of the 40th, 41st and 42nd entries are j, which implies that $j = 0$. From the 26th, 27th and 28th entries, at least two of $l + a, l + b$ and $l + h$ are 0 since $j = 0$. Hence $k = 0$ from the 19th, 20th and 21st entries. Then at least two of c, d, g are 0 from the 33th, 34th and 35th entries. This and the 12th, 13th and 14th entries imply $l = 0$. Hence at least two of a, b, h are 0 from the 19th, 20th and 21st entries. Thus we have shown (2) and (3).

By the condition (2). We have

$$\mathbf{p} = {}^t(\begin{array}{lllllll} a+g+m & b+g+m & c+h+m & d+h+m & i & e & f \mid \\ a+i+m & e+h+m & f+h+m & b+i+m & g & d & c \mid \\ d+i+m & e+g+m & f+g+m & c+i+m & a & b & h \mid \\ c+i+m & f+g+m & e+g+m & d+i+m & a & b & h \mid \\ a+i+m & f+h+m & e+h+m & b+i+m & g & c & d \mid \\ a+g+m & b+g+m & d+h+m & c+h+m & i & f & e) \end{array}$$

All the entries must be non-negative. From the entries corresponding to the square type normal disks, $a, b, c, d, e, f, g, h, i \geq 0$. If $g = 0$, then the 1st entry is $a + m$ and 2nd $b + m$. Hence we have $m \geq 0$ because $a = 0$ or $b = 0$ by the condition (3). Similarly, when $h = 0$ or $i = 0$, we can see $m \geq 0$. Thus (4)(a) follows. Therefore we can assume that $g, h, i > 0$. Then, by the condition (3) $a, b, c, d, e, f = 0$. Hence $g + m \geq 0$, $h + m \geq 0$ and $i + m \geq 0$ from the 1st, 3rd and 8th entries respectively. □

Theorem 3.3. *In the triangulated* 3*-torus, there are precisely* 11 *fundamental surfaces* $\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \mathbf{p}_d, \mathbf{p}_e, \mathbf{p}_f, \mathbf{p}_g, \mathbf{p}_h, \mathbf{p}_i, \mathbf{p}_m$ *and*
$\mathbf{p}_n = {}^t(0\,0\,0\,0\,1\,0\,0 \mid 0\,0\,0\,0\,1\,0\,0 \mid 0\,0\,0\,0\,0\,0\,1 \mid 0\,0\,0\,0\,0\,0\,1 \mid 0\,0\,0\,0\,1\,0\,0 \mid 0\,0\,0\,0\,1\,0\,0)$.

Remark 3.4. $\mathbf{p}_a$, $\mathbf{p}_c$, $\mathbf{p}_d$, $\mathbf{p}_e$, $\mathbf{p}_f$, $\mathbf{p}_h$, $\mathbf{p}_i$ are parallel to the torus which have the rectangles $ABEC$, $ABGD$, $ACFD$, $ACHG$, $ABHF$, $ADHE$, $BCFG$ as a net. $\mathbf{p}_b$ is parallel to the torus which is a union of the two triangles AFG and BCH. Recall that M, N are the middle points of the edges BG, CF. Let R, S be the middle points of the edges AD, EH respectively. $\mathbf{p}_g$ is parallel to the torus which is a union of the triangles FGR, BCS and the quandrangle $AMHN$. $\mathbf{p}_m$ is the vertex-linking sphere. $\mathbf{p}_n$ is a connected sum of 4 copies of the projective plane $\mathbb{R}P^2$, which is obtained from the torus T_i corresponding to $\mathbf{p}_i$ by a tubing operation along a subarc of the line AH connecting the two disks $T_i \cap$ (the cube).

Proof. If $m \geq 0$, then $\mathbf{p} = a\mathbf{p}_a + b\mathbf{p}_b + c\mathbf{p}_c + d\mathbf{p}_d + e\mathbf{p}_e + f\mathbf{p}_f + g\mathbf{p}_g + h\mathbf{p}_h + i\mathbf{p}_i + m\mathbf{p}_m$ with $a, b, c, d, e, f, g, h, i, m \geq 0$. Hence $\mathbf{p} = \mathbf{p}_a$, $\mathbf{p}_b$, $\mathbf{p}_c$, $\mathbf{p}_d$, $\mathbf{p}_e$, $\mathbf{p}_f$, $\mathbf{p}_g$, $\mathbf{p}_h$, $\mathbf{p}_i$ or $\mathbf{p}_m$ by Lemma 2.13.

Hence we can assume $m < 0$. Then we are in the case of (4)(b) of Lemma 3.2 rather than (4)(a). Thus
$\mathbf{p} = g\mathbf{p}_g + h\mathbf{p}_h + i\mathbf{p}_i + m\mathbf{p}_m$
$= {}^t(g+m \quad g+m \quad h+m \quad h+m \quad i \quad 0 \quad 0 \mid i+m \quad h+m \quad h+m \quad i+m \quad g \quad 0 \quad 0 \mid$
$i+m \quad g+m \quad g+m \quad i+m \quad 0 \quad 0 \quad h \mid i+m \quad g+m \quad g+m \quad i+m \quad 0 \quad 0 \quad h \mid$
$i+m \quad h+m \quad h+m \quad i+m \quad g \quad 0 \quad 0 \mid g+m \quad g+m \quad h+m \quad h+m \quad i \quad 0 \quad 0)$
Since all the entries are non-negative, $g, h, i \geq -m > 0$. When $g = 1$, $h = 1$, $i = 1$ and $m = -1$, $\mathbf{p} = \mathbf{p}_n$. Thus $\mathbf{p} \geq \mathbf{p}_n$, and hence $\mathbf{p} = \mathbf{p}_n$ by Lemma 2.14.

We can easily see that the vectors in the theorem are fundamental by considering size of vectors as in section 2. □

4. The (2,1)- and (3,1)-lens spaces and non-admissible fundamental solutions

Let τ_i be the i-th tetrahedron with vertices v_{i1}, v_{i2}, v_{i3} and v_{i4}. Glueing τ_i and τ_{i+1} along the trigons $v_{i1}v_{i2}v_{i4}$ and $v_{(i+1)1}v_{(i+1)2}v_{(i+1)3}$ for $i = 1, 2, \cdots, p$, (where $v_{(p+1)j} = v_{1j}$), we obtain a suspension of a p-gon. See Figure 8, where $p = 3$. We obatin the (p, q)-lens space, glueing the trigons $v_{i1}v_{i3}v_{i4}$ and $v_{(i+q)2}v_{(i+q)3}v_{(i+q)4}$ for $i = 1, 2, \cdots, p$.

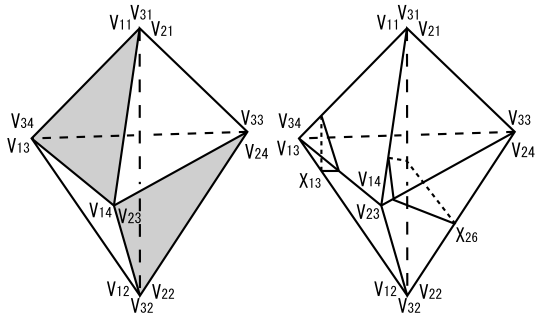

FIGURE 8. A triangulation of the $(3,1)$-lens space

For $1 \leq j \leq 4$, let X_{ij} be the normal disk of triangle type in τ_i such that it separates the vertex v_{ij} from the other vertices of τ_i. We denote by X_{i5}, X_{i6}, X_{i7} the normal disks of square type in τ_i such that X_{i5} separates the vertices v_{i1}, v_{i2} from v_{i3}, v_{i4}, X_{i6} does v_{i1}, v_{i4} from v_{i2}, v_{i3}, and X_{i7} does v_{i1}, v_{i3} from v_{i2}, v_{i4}. We array the numbers x_{ij} of the types of the normal disks X_{ij} in the order $(x_{11} \;\; x_{12} \;\; \cdots \;\; x_{17} \;\; x_{21} \;\; \cdots x_{27} \;\; \cdots x_{p1} \;\; \cdots \;\; x_{p7})$.

A non-trivial solution $\mathbf{p}$ of the matching equations is called *fundamental* if all its entries are non-negative integers and if $\mathbf{p} \neq \mathbf{q} + \mathbf{r}$ for any pair of non-trivial solutions $\mathbf{q}, \mathbf{r}$ with non-negative integer entries. It may be non-admissible, (i.e., may have two types of square normal disks in a tetrahedron), and corresponds to an immersed normal surface with self-intersection.

Similar calculation as in section 2 shows the results below.

Theorem 4.1. *For the above triangulation of the* $(2,1)$*-lens space, there are precisely* 7 *fundamental solutions* $\mathbf{p}_A, \cdots, \mathbf{p}_H$ *as below.*

$\mathbf{p}_A = {}^t(0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 0\ 0)$
$\mathbf{p}_B = {}^t(0\ 0\ 0\ 0\ 0\ 1\ 0 \mid 0\ 0\ 0\ 0\ 0\ 0\ 1)$
$\mathbf{p}_C = {}^t(0\ 0\ 0\ 0\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0\ 0\ 1\ 0)$
$\mathbf{p}_D = {}^t(1\ 1\ 0\ 0\ 0\ 0\ 0 \mid 1\ 1\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_E = {}^t(0\ 0\ 1\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 1\ 0\ 0\ 0)$
$\mathbf{p}_F = {}^t(1\ 1\ 1\ 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 1\ 1)$
$\mathbf{p}_G = {}^t(0\ 0\ 0\ 0\ 1\ 1\ 1 \mid 1\ 1\ 1\ 1\ 0\ 0\ 0)$

Remark 4.2. $\mathbf{p}_A$ is a torus surrounding the loop edge $v_{11}v_{12}$. It also surrounds the loop edge $v_{13}v_{14}$ and gives a Heegaard splitting of the $(2,1)$-lens space. $\mathbf{p}_B, \mathbf{p}_C$ are projective planes such that their double $2\mathbf{p}_B, 2\mathbf{p}_C$ are spheres surrounding the edges $v_{11}v_{14}$, $v_{21}v_{24}$ respectively. $\mathbf{p}_D, \mathbf{p}_E$ are vertex-linking spheres of v_{11}, v_{13} respectively.

Lemma 4.3. *For the above triangulation of the* $(3,1)$*-lens space, the solution space of the matching equations has a basis consisting of the* 8 *vectors below.*

$\mathbf{p}_a = {}^t(0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 0\ 0)$

$\mathbf{p}_b = {}^t(1\ 1\ 0\ 0\ 0\ 0\ 0 \mid 1\ 1\ 0\ 0\ 0\ 0\ 0 \mid 1\ 1\ 0\ 0\ 0\ 0\ 0)$
$\mathbf{p}_c = {}^t(0\ 0\ 1\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 1\ 0\ 0\ 0 \mid 0\ 0\ 1\ 1\ 0\ 0\ 0)$
$\mathbf{p}_d = {}^t(0\ 0\ 0\ 0\ 0\ 1\ 1 \mid 0\ 0\ 0\ 0\ 0\ 1\ 1 \mid 0\ 0\ 0\ 0\ 0\ 1\ 1)$
$\mathbf{p}_e = {}^t(1\ 0\ 0\ 1\ 0\ 1\ 0 \mid 0\ 1\ 1\ 0\ 0\ 1\ 0 \mid 0\ 0\ 0\ 0\ 0\ 0\ 2)$
$\mathbf{p}_f = {}^t(0\ 1\ 1\ 0\ 0\ 1\ 0 \mid 0\ 0\ 0\ 0\ 0\ 0\ 2 \mid 1\ 0\ 0\ 1\ 0\ 1\ 0)$
$\mathbf{p}_h = {}^t(1\ 0\ 0\ 1\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0\ 1\ 2\ 0 \mid 0\ 1\ 1\ 0\ 0\ 0\ 1)$
$\mathbf{p}_i = {}^t(0\ 1\ 1\ 0\ 0\ 0\ 1 \mid 1\ 0\ 0\ 1\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0\ 1\ 2\ 0)$

Remark 4.4. This basis is almost symmetric both under the 180°-rotation about the line which goes through the vertex v_{24} and the middle point of the edge $v_{13}v_{14}$, and under the reflection in the plane which contains the trigon $v_{21}v_{22}v_{24}$. These transformations only change order of entries.

Theorem 4.5. *For the above triangulation of the* $(3,1)$*-lens space, there are precisely* 6 *fundamental surfaces* $\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \mathbf{p}_e, \mathbf{p}_f$ *and* $\mathbf{p}_g$ *as below.*
$\mathbf{p}_g = {}^t(0\ 0\ 0\ 0\ 0\ 0\ 2 \mid 1\ 0\ 0\ 1\ 0\ 1\ 0 \mid 0\ 1\ 1\ 0\ 0\ 1\ 0)$

Remark 4.6. $\mathbf{p}_a$ represents the torus surrounding the loop edge $v_{11}v_{12}$. This torus surrounds also the loop edge $v_{13}v_{14}$, and gives a Heegaard splitting of the $(3,1)$-lens space. $\mathbf{p}_b, \mathbf{p}_c$ are the vertex-linking spheres of v_{11} and v_{13} respectively. $\mathbf{p}_e, \mathbf{p}_f, \mathbf{p}_g$ are spheres surrounding the edges $v_{21}v_{24}$, $v_{11}v_{14}$, $v_{31}v_{34}$ respectively.

Conjecture 4.7. For the $(p,1)$-lens space triangulated as above, the list of the fundamental surfaces is as below, where $\mathcal{E} = \{v_{11}v_{14}, v_{21}v_{24}, \cdots, v_{p1}v_{p4}\}$, a set of edges.

(1) The Heegaard splitting torus which surrounds the edge $v_{11}v_{12}$, and does also $v_{13}v_{14}$ on the other side.
(2) The vertex-linking spheres of v_{11} and v_{13}.
(3) The surrounding surface of $\cup\mathcal{E}'$ such that $\mathcal{E}' \subset \mathcal{E}$, that the edges in $\mathcal{E}'$ are less than $p/2$ and that $\mathcal{E}'$ does not contain a pair of edges $v_{i1}v_{i4}$, $v_{(i+1)1}v_{(i+1)4}$ for any $1 \le i \le p$, where $v_{(p+1)j} = v_{1j}$.
(4) The non-orientable surfaces one of which is the union of the squares X_{16}, X_{27}, X_{36}, X_{47}, $\cdots$, $X_{(p-1)6}$, X_{p7} and the other the union of X_{17}, X_{26}, X_{37}, X_{46}, $\cdots$, $X_{(p-1)7}$, X_{p6} with p even. Their doubles surround the union of the edges $v_{11}v_{14}$, $v_{31}v_{34}$, $\cdots$, $v_{(p-1)1}v_{(p-1)4}$, and that of $v_{21}v_{24}$, $v_{41}v_{44}$, $\cdots$, $v_{p1}v_{p4}$ respectively.

In the rest of this section, we show calculation of (possibly non-admissible) fundamental solutions of the matching equations for the $(3,1)$-lens space triangulated as above. The authors gave up calculating the whole cases by hand because there are too many cases. We show the calculation only in a single case.

By Lemma 4.3, we take a general solution of the matching equations as below.
$\mathbf{p} = a\mathbf{p}_a + b\mathbf{p}_b + c\mathbf{p}_c + d\mathbf{p}_d + e\mathbf{p}_e + f\mathbf{p}_f + h\mathbf{p}_h + i\mathbf{p}_i$

$$= {}^t(b+e+h \ \ b+f+i \ \ c+f+i \ \ c+e+h \ \ a \ \ d+e+f \ \ d+h+i \ \ | \\ b+i \ \ b+e \ \ c+e \ \ c+i \ \ a+h \ \ d+e+2h \ \ d+2f+i \ \ | \\ b+f \ \ b+h \ \ c+h \ \ c+f \ \ a+i \ \ d+f+2i \ \ d+2e+h)$$

If $\mathbf{p}$ is a fundamental solution other than the vectors $\mathbf{p}_a, \cdots, \mathbf{p}_f, \mathbf{p}_h, \mathbf{p}_i$ of the basis, then $\mathbf{p} \not\geq \mathbf{p}_a, \cdots, \mathbf{p}_f, \mathbf{p}_h, \mathbf{p}_i$. From the condition $\mathbf{p} \not\geq \mathbf{p}_a$, at least one of the 5th, 12th and 19th entries of $\mathbf{p}$ is 0. The condition $\mathbf{p} \not\geq \mathbf{p}_i$ implies that either at least one of the 2nd, 3rd, 7th, 8th, 11th, 14th, 19th of $\mathbf{p}$ is 0, or the 20th entry $d+f+2i$ is equal to 0 or 1. In this way, we have many cases from the other conditions and their combinations.

We consider the case where the 12th entry $a+h=0$, the 14th entry $d+2f+i=0$, the 15th entry $b+f=0$, the 18th entry $c+f=0$ and the 21st entry $d+2e+h=\epsilon$ with $\epsilon = 0$ or 1. From these conditions, $d=-2f-i$, $h=\epsilon-d-2e=\epsilon+2f+i-2e$, $a=-h=-\epsilon-2f-i+2e$, $b=-f$ and $c=-f$. Applying these conditions, we have

$$\mathbf{p} = {}^t(-e+f+i+\epsilon \ \ i \ \ i \ \ -e+f+i+\epsilon \ \ 2e-2f-i-\epsilon \ \ e-f-i \ \ -2e+i+\epsilon \ \ | \\ -f+i \ \ e-f \ \ e-f \ \ -f+i \ \ 0 \ \ -3e+2f+i+2\epsilon \ \ 0 \ \ | \\ 0 \ \ -2e+f+i+\epsilon \ \ -2e+f+i+\epsilon \ \ 0 \ \ 2e-2f-\epsilon \ \ -f+i \ \ \epsilon)$$

$= e\mathbf{p}_1 + f\mathbf{p}_2 + i\mathbf{p}_3 + \epsilon\mathbf{p}_\epsilon$, where

$\mathbf{p}_1 = {}^t(-1\ 0\ 0\ -1\ 2\ 1\ -2 \mid 0\ 1\ 1\ 0\ 0\ -3\ 0 \mid 0\ -2\ -2\ 0\ 2\ 0\ 0)$

$\mathbf{p}_2 = {}^t(1\ 0\ 0\ 1\ -2\ -1\ 0 \mid -1\ -1\ -1\ -1\ 0\ 2\ 0 \mid 0\ 1\ 1\ 0\ -2\ -1\ 0)$

$\mathbf{p}_3 = {}^t(1\ 1\ 1\ 1\ -1\ -1\ 1 \mid 1\ 0\ 0\ 1\ 0\ 1\ 0 \mid 0\ 1\ 1\ 0\ 0\ 1\ 0)$

$\mathbf{p}_\epsilon = {}^t(1\ 0\ 0\ 1\ -1\ 0\ 1 \mid 0\ 0\ 0\ 0\ 0\ 2\ 0 \mid 0\ 1\ 1\ 0\ -1\ 0\ 1)$.

Setting $\mathbf{p}_s = -\mathbf{p}_1 - \mathbf{p}_2$, $\mathbf{p}_t = -\mathbf{p}_a - 2\mathbf{p}_2 + \mathbf{p}_3$ and $\mathbf{p}_u = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$, we have

$\mathbf{p} = s\mathbf{p}_s + t\mathbf{p}_t + u\mathbf{p}_u + \epsilon\mathbf{p}_\epsilon$ with

$\mathbf{p}_s = {}^t(0\ 0\ 0\ 0\ 0\ 0\ 2 \mid 1\ 0\ 0\ 1\ 0\ 1\ 0 \mid 0\ 1\ 1\ 0\ 0\ 1\ 0)$

$\mathbf{p}_t = {}^t(0\ 1\ 1\ 0\ 1\ 0\ 3 \mid 3\ 1\ 1\ 3\ 0\ 0\ 0 \mid 0\ 1\ 1\ 0\ 2\ 3\ 0)$

$\mathbf{p}_u = {}^t(1\ 1\ 1\ 1\ -1\ -1\ -1 \mid 0\ 0\ 0\ 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0\ 0\ 0)$

$(s=-3e+2f+i, t=e-f, u=-e+f+i)$.

When $\epsilon = 0$, we can see as in section 2 that $\mathbf{p} = \mathbf{p}_s$ or $\mathbf{p}_t$. When $\epsilon = 1$, $u = 0$ or -1 from the 1st and 6th entries. We consider the case of $u=-1$, where

$$\mathbf{p} = {}^t(0 \ \ t-1 \ \ t-1 \ \ 0 \ \ t \ \ 1 \ \ 2s+3t+2 \ \ | \\ s+3t \ \ t \ \ t \ \ s+3t \ \ 0 \ \ s+2 \ \ 0 \ \ | \\ 0 \ \ s+t+1 \ \ s+t+1 \ \ 0 \ \ 2t-1 \ \ s+3t \ \ 1).$$

By Lemma 2.14, $\mathbf{p} = \mathbf{p}_t$ or $t=1$. In the latter case, we have

$\mathbf{p} = {}^t(0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 2s+5 \mid s+3\ \ 1\ \ 1\ \ s+3\ \ 0\ \ s+2\ \ 0 \mid 0\ \ s+2\ \ s+2\ \ 0\ \ 1\ \ s+3\ \ 1)$.

Again by Lemma 2.14, $\mathbf{p} = \mathbf{p}_s$ or $s=-2$. In the latter case,

$\mathbf{p} = {}^t(0\ 0\ 0\ 0\ 1\ 1\ 1 \mid 1\ 1\ 1\ 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 1\ 1)$.

Similarly, when $u=0$, $\mathbf{p} = \mathbf{p}_t, \mathbf{p}_s$ or

$\mathbf{p} = {}^t(1\ 1\ 1\ 1\ 0\ 0\ 0 \mid 1\ 1\ 1\ 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 1\ 1)$.

5. Data for other triangulations of $(4,1)$- and $(3,1)$-lens spaces

Let τ_i, v_{ij}, X_{ij} be as in the previous section 4. But we glue them in different ways in this section.

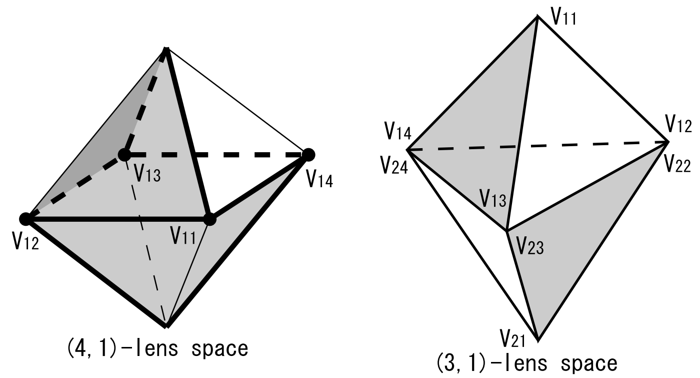

Figure 9. Other triangulations of $(4,1)$- and $(3,1)$-lens spaces

We can obtain the $(4,1)$-lens space from a single tetrahedron τ_1 by glueing the faces $v_{11}v_{12}v_{13}$ and $v_{14}v_{11}v_{12}$, and then also the faces $v_{13}v_{14}v_{11}$ and $v_{12}v_{13}v_{14}$. See Figure 9.

Theorem 5.1. *For the* $(4,1)$*-lens space triangulated as above, there are precisely* 3 *fundamental solutions* $\mathbf{p}_A = {}^t(1\ 1\ 1\ 1\ 0\ 0\ 0)$, $\mathbf{p}_B = {}^t(0\ 0\ 0\ 0\ 1\ 0\ 0)$, $\mathbf{p}_C = {}^t(0\ 0\ 0\ 0\ 0\ 1\ 1)$ *of the matching equations.*

Remark 5.2. $\mathbf{p}_A$ is the vertex-linking sphere. $\mathbf{p}_B$ is a Klein bottle.

We obtain the $(3,1)$-lens space from two tetrahedra τ_1, τ_2 by glueing $v_{12}v_{13}v_{14}$ and $v_{22}v_{23}v_{24}$, $v_{11}v_{12}v_{13}$ and $v_{21}v_{24}v_{22}$, $v_{11}v_{13}v_{14}$ and $v_{21}v_{22}v_{23}$, $v_{11}v_{14}v_{12}$ and $v_{21}v_{23}v_{24}$.

Theorem 5.3. *For the* $(3,1)$*-lens space triangulated as above, there are precisely* 11 *fundamental solutions* $\mathbf{p}_a, \cdots, \mathbf{p}_l$ *below of the matching equations.*
$\mathbf{p}_a = {}^t(1\ 0\ 0\ 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 0\ 0\ 0\ 0)$, $\mathbf{p}_b = {}^t(0\ 1\ 1\ 1\ 0\ 0\ 0 \mid 0\ 1\ 1\ 1\ 0\ 0\ 0)$,
$\mathbf{p}_c = {}^t(0\ 1\ 1\ 0\ 0\ 1\ 0 \mid 0\ 1\ 0\ 1\ 0\ 0\ 1)$, $\mathbf{p}_d = {}^t(0\ 1\ 0\ 1\ 0\ 0\ 1 \mid 0\ 0\ 1\ 1\ 1\ 0\ 0)$,
$\mathbf{p}_e = {}^t(0\ 1\ 0\ 0\ 0\ 1\ 1 \mid 0\ 0\ 0\ 1\ 1\ 0\ 1)$, $\mathbf{p}_f = {}^t(0\ 0\ 1\ 1\ 1\ 0\ 0 \mid 0\ 1\ 1\ 0\ 0\ 1\ 0)$,
$\mathbf{p}_g = {}^t(0\ 0\ 1\ 0\ 1\ 1\ 0 \mid 0\ 1\ 0\ 0\ 0\ 1\ 1)$, $\mathbf{p}_h = {}^t(0\ 0\ 0\ 1\ 1\ 0\ 1 \mid 0\ 0\ 1\ 0\ 1\ 1\ 0)$,
$\mathbf{p}_i = {}^t(0\ 0\ 0\ 0\ 1\ 1\ 1 \mid 0\ 0\ 0\ 0\ 1\ 1\ 1)$, $\mathbf{p}_j = {}^t(1\ 1\ 1\ 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 1\ 1\ 1)$,
$\mathbf{p}_k = {}^t(0\ 0\ 0\ 0\ 1\ 1\ 1 \mid 1\ 1\ 1\ 1\ 0\ 0\ 0)$

Remark 5.4. $\mathbf{p}_a$, $\mathbf{p}_b$ are the vertex-linking spheres of v_{11}, v_{12} respectively. $\mathbf{p}_c$, $\mathbf{p}_d$, $\mathbf{p}_f$ are spheres which surround the edges $v_{11}v_{14}$, $v_{11}v_{13}$, $v_{11}v_{12}$ respectively.

Each of these two triangulations of lens spaces does not have a Heegaard splitting surface in the systems of fundamental surfaces, though it contains a spine of a solid torus of the genus one Heegaard splitting. Note that it does not contain a spine of the other solid torus.

Conjecture 5.5. If a triangulation of a closed 3-manifold contains a pair of disjoint spines of the handlebodies of a Heegaard splitting, then the system of the fundamental surfaces contains a surface which gives the Heegaard splitting.

However, the above conjecture says nothing on finding a minimum genus Heegaard splitting surface.

6. Euler characteristics and the figure 8 knot

Do we really need to calculate fundamental surfaces? In this section, we show the figure 8 knot is not trivial by considering Euler characteristics of normal surfaces with respect to the well-known ideal triangulation of the knot exterior. See Figure 10.

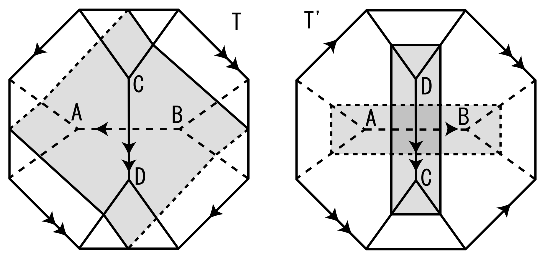

Figure 10. An ideal triangulation of the figure 8 knot exterior

In general, we can calculate Euler characteristic of a normal surface by taking the sum of contributions of its normal disks defined as below. Let Q be a normal disk in a tetrahedron. For each vertex v of Q, we set $w(v)$ to be equal to the inverse number of the multiplicity of the 1-simplex σ_1 containing v, where the multiplicity is the number of times σ_1 appears in the boundary spheres of the tetrahedra. (For example, the edge with a single arrow in Figure 10 appears 3 times in the boundary sphere of each tetrahedron. Hence the multiplicity is 6.) For each edge e of Q, we set $w(e)$ to be equal to 1 if e is contained in the boundary of the 3-manifold, and otherwise we set it to be equal to $-1/2$. Then the contribution of Q is defined to be $\chi'(Q) = \sum w(v) - \sum w(e) + 1$, where the sums are taken over all the vertices and all the edges of Q. (For example, the shaded octagon in the tetrahedron T in Figure 10 contributes by $(1/2) \times 8 - (1/2) \times 4 - 1 \times 4 + 1 = -1$.) Clearly, Euler characteristic of an embedded normal surface is the sum of the contributions of its normal disks. However, this is not true for immersed normal surfaces with self-intersection. They may go around two or more times about a 1-simplex. (For example, $\mathbf{p}_d$ in Lemma 4.3 goes twice around the axis $v_{11}v_{12}$.)

We apply this fact to the figure 8 knot exterior. In Figure 10, two truncated tetrahedra T, T' are described. As is well known, glueing the hexagonal faces A, B, C, D of T to those of T' respectively so that types and directions of arrows of the edges match, we obtain the figure 8 knot exterior E. The union of the 8 triangle faces is the boundary torus ∂E. (See, for example, page 39 in [15].) The shaded octagon in T and the two squares in T' together form the genus one Seifert surface S of the figure 8 knot. (This can be obtained from the generator $[A+B] \in H_2(E, \partial E)$ by cutting and pasting near the doubly arrowed edge and slightly isotoping the resulting surface.) The boundary circle ∂S intersects each trigonal faces precisely in a single normal arc.

If the knot is trivial, then this circle ∂S bounds a disk in E, and we can easily show that there is a normal surface R bounded by ∂S such that R is an essential disk in E. We can show easily that there is no such disk in E to see that the figure 8 knot is not trivial. A truncated tetrahedron has great many types of normal disks. (See section 2 for the definition of normal disks.) A normal disk Q is called a m:n-disk if it has m edges in trigons and n edges in hexagons. Then $\chi'(Q) = -m/6 - n/3 + 1$. Only 1:2-disk contribute by positive value 1/6, and the other normal disks do by 0 or negative values. Since a disk is of Euler characteristic 1, we need at least six 1:2-disks. However, this is impossible. If one of T and T' contains three or four 1:2-disks, then the other does at most one. See Figure 11, where we can find four 1 : 2-disks in T and normal arcs corresponding their edges in T'. This completes the proof.

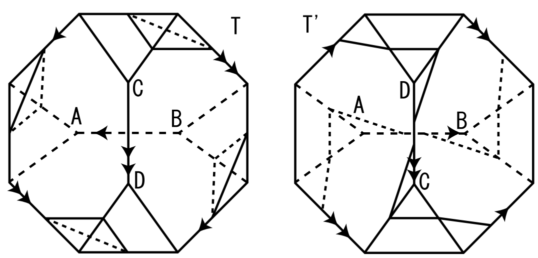

FIGURE 11. 1:2-disks in T and corresponding normal arcs in T'

Remark 6.1. The authors expect that this method works for a triangulation of a knot exterior such that every type of normal disk with positive contribution to Euler characteristic has an edge contained in the boundary torus.

Problem 6.2. Does this method have some relation with the average edge order? See for example [14].

A. APPENDIX

```
(* This program runs on Mathematica 5.0, *)
(* calculating the vertex solutions of a system of matching equations. *)
(* This takes about 5 minutes for 3 tetrahedra, *)
(* 2hours for 4 tetrahedra and several days for 5 tetrahedra. *)
(* First, we form the coefficient matrix a. *)
```

```
{
 (* t is the number of tetrahedra. *)
 t = 3;
 dim = 7t;
 a = Table[0, {dim + 1}, {dim}];
 (* The entries equal to 1. *)
 a = ReplacePart[a, 1,
    {{1, 1}, {1, 5}, {2, 2}, {2, 6}, {3, 3}, {3, 7},
     {4, 3}, {4, 6}, {5, 2}, {5, 7}, {6, 4}, {6, 5},
     {7, 11}, {7, 13}, {8, 10}, {8, 14}, {9, 9}, {9, 12}
    }];
 (* The entries equal to - 1. *)
 a = ReplacePart[a, -1,
    {{1, 16}, {1, 20}, {2, 15}, {2, 19}, {3, 17}, {3, 21},
     {4, 8}, {4, 13}, {5, 9}, {5, 14}, {6, 10}, {6, 12},
     {7, 17}, {7, 20}, {8, 16}, {8, 21}, {9, 18}, {9, 19}
    }];
 TableForm[a]
}

(* Calculation of all the possibly non - admissible vertex solutions. *)
(* The program below works for any system of matching equations. *)
{
 vertex = {};
 b = RowReduce[a];
 (* Print a basis of the solution space of the matching equations. *)
 Print[NullSpace[b]];
 soldim = Length[NullSpace[b]];
 reddim = soldim - 1;
 (* c is a list of matrices, the number of which is equal to reddim. *)
 (* The first one is b. *)
 c = Array[0, Prepend[Dimensions[b], reddim]];
 c = ReplacePart[c, b, 1];
 (* dims is the list of dimensions of the kernels of the matrices of c. *)
 dims = Table[0, {soldim}];
 dims = ReplacePart[dims, soldim, 1];
 (* We set the second matrix of c to be one obtained from b *)
 (* by adding a fundamental vector as the last row. *)
```

```
 (* Then we reduce it by elementary transformation on rows. *)
 (* If the dimension of the kernel did not decrease, *)
 (* we change the added fundamental vector to another one. *)
 (* If the dimension of the kernel decreased, *)
 (* then we add another fundamental vector and set it as the 3rd matrix. *)
 (* We repeat this operation till dimension = 1. *)
 kaname[depth_, tail_] :=
  Do[{If[x == 1, c = ReplacePart[c, 1, {depth, 7t + 1, tail + 1}],
               {c = ReplacePart[c, 0, {depth, 7t + 1, tail + x - 1}];
                c = ReplacePart[c, 1, {depth, 7t + 1, tail + x}]}];
                sol = NullSpace[c[[depth]]];
      dims = ReplacePart[dims, Length[sol], depth + 1];
      If[dims[[depth + 1]] < dims[[depth]],
         If[depth == reddim,
             {r = sol[[1]];
              (* Multiply r so that the entries become integer. *)
              Do[{vec = z*r;
                  If[Count[vec, _Integer] == dim, {r = vec; Break[]}]},
              {z, 100}];
              (* We neglect r with positive entries and negative ones. *)
              (* Otherwise, we print r if it has not appeared yet. *)
              If[Apply[Plus, Map[Abs, r]] == Abs[Apply[Plus, r]],
                 If[FreeQ[vertex, r], {vertex = Append[vertex, r];
                                       Print[r]}
                 ]
              ]},
              (* If the kernel is not of dimension 1, *)
              (* we add another fundamental vector. *)
              {c = ReplacePart[c, RowReduce[c[[depth]]], depth + 1];
               kaname[depth + 1, tail + x] }
         ]
     ];
     If[x == dim - tail - (reddim - depth),
         c = ReplacePart[c, 0, {depth, 7t + 1, tail + x}]]},
  {x, dim - tail - (reddim - depth)}];
 kaname[1, 0]
}
```

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