Q-FUNDAMENTAL SURFACES IN LENS SPACES

MIWA IWAKURA AND CHUICHIRO HAYASHI

Abstract. We determine all the Q-fundamental surfaces in $(p,1)$-lens spaces and $(p,2)$-lens spaces with respect to natural triangulations with p tetrahedra. For general (p,q)-lens spaces, we give an upper bound for elements of vectors which represent Q-fundamental surfaces with no quadrilateral normal disks disjoint from the core circles of lens spaces. We also give some examples of non-orientable closed surfaces which are Q-fundamental surfaces with such quadrilateral normal disks.

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1. Introduction

In [11], K. Kneser introduced normal surfaces which are in "beautiful" position with respect to a triangulation of a 3-manifold, to show the existence of prime decomposition. A normal surface intersects each tetrahedron of a triangulation in a disjoint union of trigons and quadrilaterals.

In [5], W. Haken found normal surfaces correspond to non-negative solutions of a certain system of simultaneous linear equations with integer coefficients, called the matching equations. He introduced fundamental surfaces, to obtain an algorithm to decide if a given knot is trivial or not. The set of fundamental solutions forms the Hilbert basis of the space of non-negative integer solutions. There are only finite number of fundamental solutions for each system. Since almost all sorts of important surfaces, such as essential spheres, essential tori, essential disk, essential annuli, knot spanning surfaces with minimal Euler characteristics and so on, can be deformed to fundamental surfaces, we obtain principal algorithms for 3-manifold topology. See, for example, [9] and [15]. The theory of fundamental surfaces was also applied to problems other than algorithms. See [6] and [12], for example.

There is an algorithm which determines all the fundamental surfaces for a given triangulation of a 3-manifold. However, it is not practical because the numbers of variables and equations of the matching equations are very large even for simplest 3-manifolds. Variables of the matching equations correspond to types of normal disks. There are 4 types of trigonal normal disks and 3 types of quadrilateral normal disks in each tetrahedron as shown in

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Figure 2. Hence the number of variables is $7t$ if a triangulation has t tetrahedra. When a closed 3-manifold with a triangulation T is given, three matching equations arise for every face of T, and hence the number of equations is $3 \times 4t/2 = 6t$. The Q-theory introduced by J. Tollefson in [16] uses the Q-matching equations with only $3t$ variables corresponding to the types of quadrilaterals. The number of equations is equal to the number of edges of the triangulation. However, Q-theory does not take care of Euler characteristic. Incompressible surfaces can be deformed into Q-fundamental surfaces. But, the resulting surfaces may have smaller Euler characteristics than the original ones.

In practice, normal surfaces in the figure eight knot complement with an ideal triangulation are studied very well. See [1], [10], [13] and [14]. In [4], E. A. Fominykh gives a complete description of the fundamental surfaces for infinite series of 3-manifolds including (p, q)-lens spaces with certain handle decompositions (not with triangulations). His argument is geometrical. Haken's original normal surface theory is based on handle decompositions rather than triangulations. However, triangulation is easier to be treated by a computer. L. B. Treybig gave an exposition of normal surface theory based on triangulations in [17].

In this paper, we determine all the Q-fundamental surfaces for $(p, 1)$-lens spaces and $(p, 2)$-lens spaces naturally triangulated with p tetrahedra by algebraic calculations. For general (p, q)-lens spaces, we give an upper bound for elements of vectors which represent Q-fundamental surfaces with no quadrilateral normal disks disjoint from the core circles of lens spaces. On the other hand, in the last section we also give an example of non-orientable closed surface of maximal Euler characteristic which is fundamental and has three parallel copies of such quadrilateral normal disks. In fact, there is an infinite sequence of (p_n, q_n)-lens spaces ($n \geq 2$) which contain fundamental surfaces homeomorphic to the connected sum of n projective planes with $n - 2$ parallel sheets of a normal disk as above (the detail will be given in [7]). In lens spaces, non-orientable closed surfaces with maximum Euler characteristics are interesting. A formula for calculating the maximum Euler characteristic is given by G. E. Bredon and J. W. Wood in [2].

B. A. Burton established a computer program *Regina* which determines all the "vertex surfaces" with respect to the (Q-)matching equations for a given triangulated 3-manifold. See [3]. The authors don't know whether there is a computer software which determines all the fundamental surfaces for general triangulated 3-manifolds or not. Vertex surfaces are introduced in [9] by W. Jaco and J. Tollefson. We can calculate them by computer much more easily than fundamental ones. Most important two-sided surfaces can be deformed into vertex surfaces. However, non-orientable surfaces in lens spaces are one-sided. In fact, we show later that the non-orientable closed surface with maximum Euler characteristic in the $(p, 1)$-lens space can be represented by a Q-fundamental surface, but cannot be by a normal surface corresponding to a vertex solution.

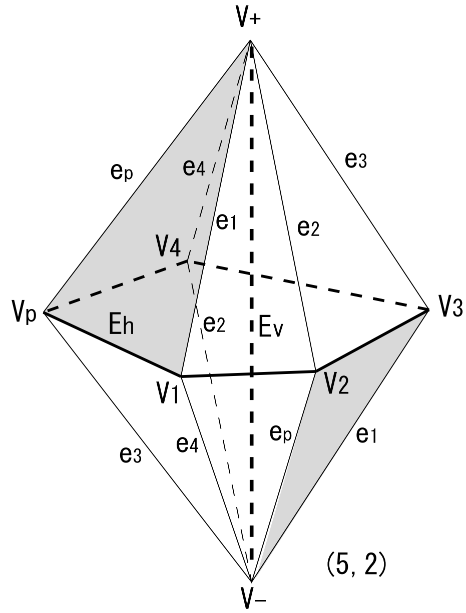

FIGURE 1

We can obtain a (p, q)-*lens space* from a suspension of a p-gon by gluing each trigonal face in the upper hemisphere with that in the lower hemisphere, performing $(2\pi q/p)$-rotation and taking a mirror image about the equator. See Figure 1. Precisely, the trigon $v_+v_iv_{i+1}$ is glued to $v_-v_{i+q}v_{i+q+1}$, where indices are considered in modulo p. The edge e_i connects v_+ and v_i, and also v_- and v_{i+q}. The horizontal edges are all glued up together into an edge E_h. Taking an axis E_v connecting the vertices v_+ and v_- in the suspension, we can decompose it into p tetrahedra. This gives a natural triangulation $T(p, q)$ of a (p, q)-lens space. The i-th tetrahedron τ_i has vertices v_+, v_-, v_i and v_{i+1}.

We recall the definition of Q-fundamental surfaces. Let M be a closed 3-manifold and T a triangulation of M, that is, a decomposition of M into finitely many tetrahedra. Let F be a closed surface embedded in M. F is called a *normal surface* with respect to the triangulation T if F intersects each tetrahedron τ in disjoint union of normal disks, or in the empty set. There are two kinds of *normal disks*, trigons called T-disks, and quadrilaterals called Q-disks. Each tetrahedron contains 4 types of T-disks and 3 types of Q-disks as in illustrated in Figure 2.

If two or three types of Q-disks exist in a single tetrahedron, then they intersect each other. See Figure 3 (1). Hence any normal surface intersects each tetrahedron in Q-disks of the same type and T-disks (Figure 3 (2)). This is called *the square condition*.

Tollefson introduced Q-coordinates representing normal surfaces in [16]. We first number types of Q-disks. In our case, we number the 3-types of Q-disks $i1$, $i2$ and $i3$ in τ_i as in Figure 4, where the axis E_v is in front of the tetrahedron. X_{i1} separates E_h and E_v, X_{i2}

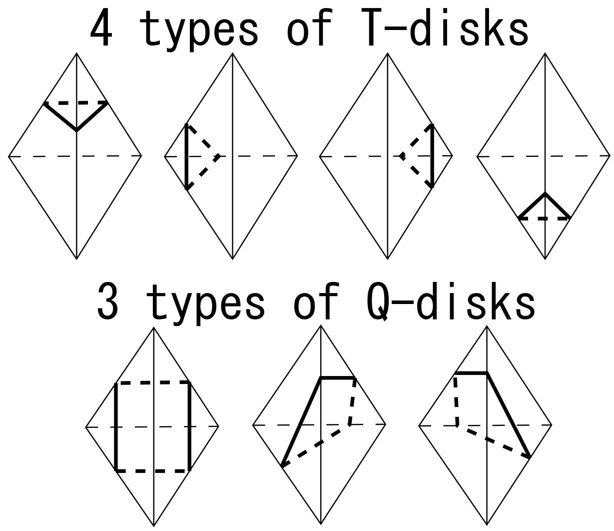

FIGURE 2

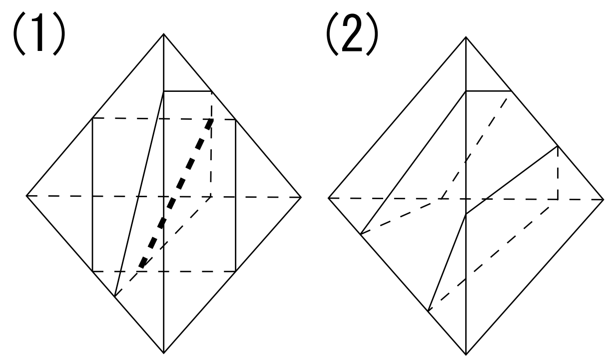

FIGURE 3

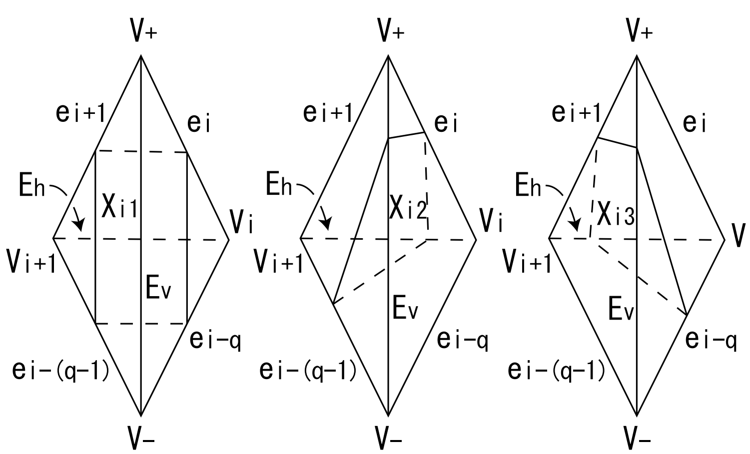

FIGURE 4

separates e_{i+1} and e_{i-q} and X_{i3} does e_i and $e_{i-(q-1)}$. Let x_{ij} be the number of Q-disks of type ij contained in a normal surface F. Then we place them in a vertical line in the order of indices lexicographically, to obtain a *Q-coordinate* of F.

Tollefson found the Q-coordinate of a normal surface satisfies the linear system of the Q-matching equations as below. One Q-matching equation arises from each edge of the triangulation T under consideration. Let e be an edge of T. We observe each tetrahedron containing e, placing e in front. Then the sense of a Q-disk with respect to e is $0, +1$ or -1 if it is flat, left side up or right side up respectively. See Figure 5. The Q-matching equation about e derives from the constraint that the number of Q-disks with plus sense is equal to that with minus sense around the edge e. See Figure 6.

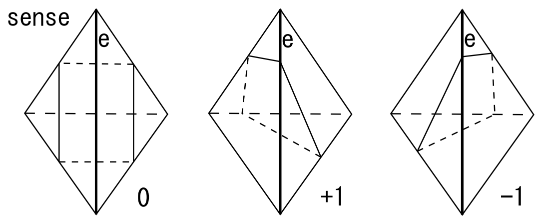

FIGURE 5

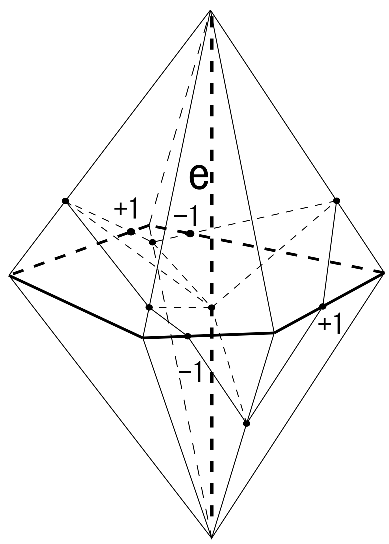

FIGURE 6

To define Q-fundamental surface, we need some terminologies on algebra. Let $\mathbf{v} = {}^t(v_1, \cdots, w_n)$, $\mathbf{w} = {}^t(w_1, \cdots, w_n)$ be vectors in $\mathbb{R}^n$, where ${}^t\mathbf{x}$ denotes the transposition of $\mathbf{x}$. We write $\mathbf{v} \leq \mathbf{w}$ if $v_i \leq w_i$ for all $i \in \{1, \cdots, n\}$. $\mathbf{v} < \mathbf{w}$ means that both $\mathbf{v} \leq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$ hold. A vector $\mathbf{u} \in \mathbb{R}^n$ is *non-negative* if $\mathbf{0} \leq \mathbf{u}$, and *integral* if all its elements are in $\mathbb{Z}$.

We consider *Q-matching equations* for all the edges, to obtain a linear system of equations. Tollefson showed that a non-zero non-negative integral solution of the Q-matching equations determines a unique normal surface with no trivial component if it satisfies the square condition, where a trivial component is a normal surface composed of T-disks and containing no Q-disks, that is, a 2-sphere surrounding a vertex of T.

Let $A\mathbf{x} = \mathbf{0}$ be a linear system of equations, where A is a matrix with all the elements in $\mathbb{Z}$, and $\mathbf{x}$ is a vector of variables. V_A denotes the solution space of the linear system considered in $\mathbb{R}^n$. A non-zero non-negative integral solution $\mathbf{v}$ is called a *fundamental solution*, if there are no integral solution $\mathbf{v}' \in V_A$ with $\mathbf{0} < \mathbf{v}' < \mathbf{v}$. A non-zero non-negative integral solution $\mathbf{v}$ is called a *vertex solution*, if for every positive integer k, all the integral solutions $\mathbf{v}''$ with $\mathbf{0} \leq \mathbf{v}'' \leq k\mathbf{v}$ are multiples of $\mathbf{v}$. This condition coincides with that for fundamental solutions when $k = 1$. Hence a vertex solution is a fundamental solution.

A normal surface is called a *Q-fundamental surface* if it corresponds to a fundamental solution of the system of the Q-matching equations respectively. A two sided normal surface is called a *vertex surface* if its coordinate is either a vertex solution of the Q-maching equations or twice a vertex solution representing a one sided surface. Tollefson showed that if an irreducible ∂-irreducible triangulated 3-manifold contains a two sided incompressible ∂-incompressible surface, then there is such one which is Q-vertex. Note that the set of fundamental surfaces with respect to Haken's matching equations contains such a surface with maximum Euler characteristic, while the set of the Q-fundamental surfaces may not.

For non-orientable closed surfaces with maximal Euler characteristics in lens spaces, we can apply the next theorem, since non-orientable surfaces in orientable 3-manifolds are one sided, and hence non-separating. Note that a non-separating surface in a (p, q)-lens space M must be non-orientable because the 1-dimensional homology $H_1(M; \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.

Theorem 1.1. *Let M be a closed 3-manifold, T a triangulation of M. Suppose that M contains a non-separating closed surface F. Then M contains such one which is Q-fundamental, and such one with maximum Euler characteristic which is fundamental with respect to T.*

In the above theorem, M is orientable or non-orientable, and F is one sided or two sided. To establish a result of this type on one sided surface in order to obtain a one sided fundamental surface seems to be almost impossible since a Haken sum on two sided surfaces may yield a one sided surface as in Figure 7.

In our situation of (p, q)-lens space, the $(3i-2)$-nd, $(3i-1)$-st and $3i$-th elements of a Q-coordinate together form the i-th *block* for $1 \leq i \leq p$. These elements correspond to the numbers of quadrilateral disks of type X_{i1}, X_{i2}, X_{i3} in τ_i. We often put a vertical line "|" instead of a comma between every adjacent pair of blocks.

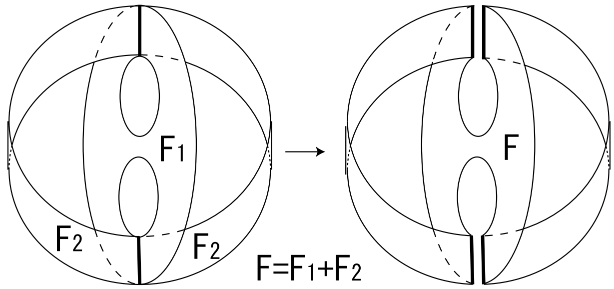

FIGURE 7

Theorem 1.2. (i) *The triangulation $T(2,1)$ of $(2,1)$-lens space has exactly three Q-fundamental surfaces* $\mathbf{f}_1 = {}^t(1,0,0 \mid 1,0,0)$, $\mathbf{f}_2 = {}^t(0,1,0 \mid 0,0,1)$, $\mathbf{f}_3 = {}^t(0,0,1 \mid 0,1,0)$.

$\mathbf{f}_1$ *represents a Heegaard splitting torus which surrounds the core circle E_v. It does E_h also on the other side of it. Each of $\mathbf{f}_2$ and $\mathbf{f}_3$ represents a projective plane. π-rotation about the axis E_v carries $\mathbf{f}_3$ to $\mathbf{f}_2$. $2\mathbf{f}_2$ represents the inessential 2-sphere surrounding the edge e_2, and $2\mathbf{f}_3$ such one surrounding e_1.*

(ii) *The triangulation $T(p,1)$ of $(p,1)$-lens space with $p \geq 3$ has exactly $p+3$ Q-fundamental surfaces $\mathbf{t}'_1, \cdots, \mathbf{t}'_p, \mathbf{f}'_1, \mathbf{f}'_2$ and $\mathbf{f}'_3$ below when p is even, and $p+1$ Q-fundamental surfaces $\mathbf{t}'_i$'s and $\mathbf{f}'_1$ when p is odd.*

$\mathbf{t}'_i = {}^t(0,0,0 \mid \cdots \mid 0,0,0 \mid 0,1,0 \mid 0,0,2 \mid 0,1,0 \mid 0,0,0 \mid \cdots \mid 0,0,0)$

$\mathbf{f}'_1 = {}^t(1,0,0 \mid 1,0,0 \mid \cdots \mid 1,0,0)$

$\mathbf{f}'_2 = {}^t(0,1,0 \mid 0,0,1 \mid 0,1,0 \mid 0,0,1 \mid \cdots \mid 0,1,0 \mid 0,0,1)$

$\mathbf{f}'_3 = {}^t(0,0,1 \mid 0,1,0 \mid 0,0,1 \mid 0,1,0 \mid \cdots \mid 0,0,1 \mid 0,1,0)$

$\mathbf{t}'_i$ *has ${}^t(0\ 0\ 2)$ as the i-th block, and represents the inessential 2-sphere surrounding the edge e_i. $\mathbf{f}'_2$ and $\mathbf{f}'_3$ represent a non-orientable closed surface with maximum Euler characteristic, which is the connected sum of $p/2$ projective planes. Note that $2\pi/p$-rotation about the axis E_v carries $\mathbf{f}'_3$ to $\mathbf{f}'_2$. $\mathbf{f}'_2$ and $\mathbf{f}'_3$ are not vertex solutions of the Q-mathing equations since $2\mathbf{f}'_2 = \mathbf{t}'_2 + \mathbf{t}'_4 + \cdots + \mathbf{t}'_p$ and $2\mathbf{f}'_3 = \mathbf{t}'_1 + \mathbf{t}'_3 + \cdots + \mathbf{t}'_{p-1}$. $\mathbf{f}'_1$ represents a Heegaard splitting torus which surrounds the core circles E_v and E_h.*

(iii) *Let p be an odd integer with $p \geq 5$. The triangulation $T(p,2)$ of $(p,2)$-lens space has exactly $p+1$ Q-fundamental surfaces*

$\mathbf{t}''_i = {}^t(0,0,0 \mid \cdots \mid 0,0,0 \mid 0,1,0 \mid 0,0,1 \mid 0,0,1 \mid 0,1,0 \mid 0,0,0 \mid \cdots \mid 0,0,0)$,

$\mathbf{f}''_1 = {}^t(1,0,0 \mid 1,0,0 \mid \cdots \mid 1,0,0)$.

$\mathbf{t}''_i$ *with the $(i-1)$-th and the $(i+2)$-th blocks being ${}^t(0\ 1\ 0)$ represents the 2-sphere surrounding the edge e_i. $\mathbf{f}''_1$ represents the Heegaard splitting torus surrounding E_v and E_h.*

We consider a general (p,q)-lens space. Since the (p,q)-lens space is homeomorphic to the $(p,p-q)$-lens space, it is sufficient to consider the case of $2 \leq q < p/2$.

Lemma 1.3. *For the triangulation $T(p,q)$ of the (p,q)-lens space with $p \geq 5$, $2 \leq q < p/2$ and $GCM(p,q) = 1$, the vectors $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$, $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p$ as below form a basis of the solution space in $\mathbb{R}^{3t}$ of the Q-matching equations.*

$$(\textit{The } j\textit{-th block of } \mathbf{s}_i) = \begin{cases} {}^t(1,1,1) & \text{if } j = i \\ {}^t(0,0,0) & \text{otherwise} \end{cases}$$

$$(\textit{The } j\textit{-th block of } \mathbf{t}_i) = \begin{cases} {}^t(0,1,0) & \text{if } j = i-1 \text{ or } i+q \\ {}^t(0,0,1) & \text{if } j = i \text{ or } i+q-1 \\ {}^t(0,0,0) & \text{otherwise} \end{cases}$$

Hence a general solution $\mathbf{v}$ in $\mathbb{R}^{3t}$ is presented as below.

$$\begin{aligned} \mathbf{v} &= a_1\mathbf{s}_1 + \cdots + a_p\mathbf{s}_p + b_1\mathbf{t}_1 + \cdots + b_p\mathbf{t}_p \\ &= {}^t(a_1,\ a_1 + b_2 + b_{p-q+1},\ a_1 + b_1 + b_{p-q+2} \mid \cdots \mid a_i,\ a_i + b_{i+1} + b_{p-q+i},\ a_i + b_i + b_{p-q+i+1} \mid \cdots \\ &\qquad \cdots \mid a_p,\ a_p + b_1 + b_{p-q},\ a_p + b_p + b_{p-q+1}), \end{aligned}$$

with $a_1, \cdots, a_p, b_1, \cdots, b_p \in \mathbb{R}$.

We set $\mathcal{B} = \{b_1, b_2, \cdots, b_p\}$, and also $\mathcal{B}_0 = \{b_2, b_4, \cdots, b_p\}$, $\mathcal{B}_1 = \{b_1, b_3, \cdots, b_{p-1}\}$ when p is even.

Lemma 1.4. *Suppose that $\mathbf{v} = a_1\mathbf{s}_1 + \cdots + a_p\mathbf{s}_p + b_1\mathbf{t}_1 + \cdots + b_p\mathbf{t}_p$ is contained in $\mathbb{Z}^{3p}$. Then $a_k \subset \mathbb{Z}$ for all integer k with $1 \leq k \leq p$, and either $\mathcal{B}_x \subset \mathbb{Z}$ or $\mathcal{B}_x \subset \mathbb{Z} + 1/2$ holds for $\mathcal{B}_x = \mathcal{B}$, $\mathcal{B}_0$ and $\mathcal{B}_1$.*

Theorem 1.5. *Assume that $\mathbf{v} = a_1\mathbf{s}_1 + \cdots + a_p\mathbf{s}_p + b_1\mathbf{t}_1 + \cdots + b_p\mathbf{t}_p$ ($a_1, \cdots, a_p, b_1, \cdots, b_p \in \mathbb{R}$) represents a Q-fundamental solution, and that $a_k = 0$ for all k.*

- (I) *When p is even, one of $\mathcal{B}_0$ and $\mathcal{B}_1$ is contained in $\mathbb{Z}$ or $\mathbb{Z} + 1/2$ and the other is $\{0\}$.*
- (II) *Suppose either p is odd and $\mathcal{B} \subset \mathbb{Z}+1/2$, or p is even and one of $\mathcal{B}_0$ and $\mathcal{B}_1$ is contained in $\mathbb{Z} + 1/2$ and the other is $\{0\}$. Then $b_i \in \{-1/2, 0, 1/2\}$ for $\forall i \in \{1, \cdots, p\}$.*
- (III) *Suppose either p is odd and $\mathcal{B} \subset \mathbb{Z}$, or p is even and one of $\mathcal{B}_0$ and $\mathcal{B}_1$ is contained in $\mathbb{Z}$ and other is $\{0\}$*!%*Then $b_i \in \{-1, 0, 1\}$ for $\forall i \in \{1, \cdots, p\}$.*

Theorem 1.6. *If p is odd, $q \geq 2$, $\mathcal{B} \subset \mathbb{Z} + 1/2$ and $a_k = 0$ for $\forall k \in \{1, \cdots, p\}$, then $\mathbf{v} = a_1\mathbf{s}_1 + \cdots + a_p\mathbf{s}_p + b_1\mathbf{t}_1 + \cdots + b_p\mathbf{t}_p$ cannot be a Q-fundamental solution.*

Lemma 1.7. *Let F be a normal surface in the (p,q)-lens space with the triangulation $T(p,q)$. If F intersects each of E_v and E_h in a single point and contains a normal disk of type X_{k2} or X_{k3}, then F is fundamental with respect to Haken's matching equations.*

When p is even and $q \geq 3$, in the (p,q)-lens space the non-orientable closed surface represented by the normal surface $\mathbf{h} = (\sum_{k=1}^{p/2} \mathbf{t}_{2k-1})/2 = {}^t(0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 1\ 0|\cdots|0\ 0\ 1|0\ 1\ 0)$ is fundamental by Lemma 1.7. However, it is not Q-fundamental because it is larger than $\mathbf{t}_i$ in Lemma 1.3. In fact, it is not of maximal Euler characteristic. We see examples of compressing disks for the surface represented by $\mathbf{h}$ in the last section.

The condition $a_k = 0$ for $\forall k \in \{1, \cdots, p\}$ is very strong. Q-fundamental surfaces with some a_k being non-zero exist. We show several examples of such Q-fundamental surfaces in the last section. Some of them are non-orientable closed surfaces with maximal Euler characteristics, and one of them has three parallel sheets of a normal disk of type X_{k1}.

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2. Proof of Theorem 1.1

Let F be a (possibly disconnected) surface in a 3-manifold M. We say F *separates* M or F is *separating* in M if there are submanifolds M_+ and M_- of M such that (1) both M_+ and M_- are unions of components of $M - F$, (2) $M_+ \cup M_- = M - F$, (3) $M_+ \cap M_- = \emptyset$ and (4) $F = \text{closure}\,(M_+) \cap \text{closure}\,(M_-)$. If all the components of F are separating in M, then F is separating. Hence F contains a component which is non-separating in M when F is non-separating. However, the converse is not true, a disjoint union of two parallel copies of a non-separating two sided surface is separating.

Let M be a compact 3-manifold possibly with boundary, and F a surface properly embedded in M. We say F is *geometrically compressible* if there is an embedded disk D, called a *compressing disk*, in M with $D \cap F = \partial D$ such that the circle ∂D does not bound a disk in F. Otherwise, F is *geometrically incompressible*.

Suppose that a triangulation T of M is given. Then, as is well-known, a geometrically incompressible surface can be isotoped so that it is deformed into a normal surface with respect to T. We can apply the next two lemmas to a non-orientable closed surface with maximal Euler characteristic in a lens space. Note that F is one sided or two sided in M in the next two lemmas.

Lemma 2.1. *Let M be a compact 3-manifold (possibly with boundary), and F a non-separating surface properly embedded in M. If F is of maximum Euler characteristic among all the non-separating surfaces properly embedded in M, then F is geometrically incompressible.*

Proof. Suppose, for a contradiction, that F is geometrically compressible. We perform a compressing operation on F along a compressing disk D. That is, we take a tubular neighbourhood $N(D) \cong D \times I$ of D so that $N(D) \cap F = (\partial D) \times I$, and set $F' = (F - ((\partial D) \times I)) \cup (D \times \partial I)$. The resulting compressed surface F' is of larger Euler characteristic than F by two. In fact, F' is non-separating. Suppose not. Then F' separates M into two regions M'_+ and M'_-. Without loss of generality, we can assume that $N(D) \subset \text{closure}\,(M'_+)$. Then F separates M into two regions $M'_+ - N(D)$ and $\text{int}(N(D) \cup M'_-)$. This is a contradiction. □

Lemma 2.2. *Let M be a compact 3-manifold (possibly with boundary), and T a triangulation of M. If $F = F_1 + F_2$ where F, F_1, F_2 are (possibly disconnected) surfaces that are normal with respect to T and $F_1 + F_2$ denotes the Haken sum, and if F is non-separating in M, then either F_1 or F_2 is non-separating.*

Proof. Suppose, for a contradiction, that F_i separates M into two submanifolds M_i^+ and M_i^- for $i = 1$ and 2. Each switch along an intersection curve of $F_1 \cap F_2$ joins either $M_1^+ \cap M_2^+$ and $M_1^- \cap M_2^-$ or $M_1^+ \cap M_2^-$ and $M_1^- \cap M_2^+$. Hence F separates M into the two regions $(M_1^+ \cap M_2^+) \cup (M_1^- \cap M_2^-)$ and $(M_1^+ \cap M_2^-) \cup (M_1^- \cap M_2^+)$. This is a contradiction. □

The next Propositions 2.3 and 2.4 show Theorem 1.1.

Proposition 2.3. *Let M be a closed 3-manifold with a triangulation T. If M contains a non-separating closed surface F, then it contains a non-separating closed surface F' such that F' is fundamental with respect to T and has Euler characteristic $\chi(F') \geq \chi(F)$.*

Proof. We use Lemma 2.1 in [8]. See section 2 of [8] for definitions of terminologies such as reduced form, patches and so on. Though handle decompositions of 3-manifolds are considered in [8], the arguments in the proof of Lemma 2.1 are valid also for triangulations.

Let F'' be a non-separating surface in M such that it has maximal Euler characteristic among all the non-separating surfaces in M. Then F'' is geometrically incompressible by Lemma 2.1. Hence an adequate isotopy deforms F'' to a normal surface with respect to T. Among all the non-separating surfaces with maximal Euler characteristic which are normal with respect to T, let F' be one with $w(F') = |F' \cap T^{(1)}|$ minimal, where $|F' \cap T^{(1)}|$ is the number of intersection points of F' and the 1-skelton of T.

We will show that F' is fundamental. Suppose, for a contradiction, that F' is not. Then there are normal surfaces F_1, F_2 with $F' = F_1 + F_2$. We can assume, without loss of generality, that this Haken sum is in reduced form. By Lemma 2.1 in [8], the Haken sum $F_1 + F_2$ has no disk patches. Then both F_1 and F_2 has Euler characteristics larger than or equal to that of F'. Moreover, either F_1 or F_2, say F_1 is non-separating by Lemma 2.2. Since $w(F') = w(F_1) + w(F_2) > w(F_1)$, we obtain a contradiction to the minimality of $w(F)$. □

Proposition 2.4. *Let M be a closed 3-manifold with a triangulation T. If M contains a non-separating closed surface F, then it contains a non-separating closed surface F' such that F' is Q-fundamental with respect to T.*

Proof. We assume that the readers have good familiality with the paper [16].

As in the proof of Proposition 2.3, M contains a normal non-separating closed surface. Let F' be one with minimal number of Q-disks among all the non-separating surfaces which are normal. We will show that F' is Q-fundamental.

Assume, for a contradiction, that F' is not Q-fundamental. Then there are non-trivial normal surfaces F_1, F_2 and a union of trivial normal surface Σ such that $F' + \Sigma = F_1 + F_2$. (Recall that a normal surface is called trivial if it consists of trigonal normal disks and has no quadrilateral normal disks, and that a normal surface is determined by a Q-coordinate up to trivial components.) Since F' is non-separating, $F' + \Sigma = F' \sqcup \Sigma$ is also non-separating in M. If F_1 and F_2 are both separating, then Lemma 2.2 shows that $F' + \Sigma$ is separating, which is a contradiction. Hence F_1 or F_2, say F_1 is non-separating, and has a non-separating component F^*. However, F^* has smaller or equal number of Q-disks than F_1, and F_1 has strictly smaller number of Q-disks than $F_1 + F_2 = F + \Sigma$. This is a contradiction. $\square$

3. The (2,1)-lens space

In this section we prove Theorem 1.2 (1). For the triangulation $T(2,1)$ of the $(2,1)$-lens space, the senses of the types the quadrilateral disks X_{k1}, X_{k2}, X_{k3} in τ_i are as below, where $\{j,k\} = \{1,2\}$ and $\epsilon_{k,ki}$, $\epsilon_{j,ki}$, $\epsilon_{E_h,ki}$, $\epsilon_{E_V,ki}$ denote the senses of X_{ki} with respect to the edges e_k, e_j, E_h, E_v respectively.

$\epsilon_{k,k1} = -2,\quad \epsilon_{k,k2} = 2,\quad \epsilon_{k,k3} = 0,$
$\epsilon_{j,k1} = 2,\quad \epsilon_{j,k2} = 0,\quad \epsilon_{j,k3} = -2,$
$\epsilon_{E_h,k1} = 0,\quad \epsilon_{E_h,k2} = -1,\quad \epsilon_{E_h,k3} = 1,$
$\epsilon_{E_v,k1} = 0,\quad \epsilon_{E_v,k2} = -1,\quad \epsilon_{E_v,k3} = 1$

Then the coefficient matrix of the Q-matching equations is as below.

$$\left(\begin{array}{ccc|ccc} \epsilon_{1,11} & \epsilon_{1,12} & \epsilon_{1,13} & \epsilon_{1,21} & \epsilon_{1,22} & \epsilon_{1,23} \\ \epsilon_{2,11} & \epsilon_{2,12} & \epsilon_{2,13} & \epsilon_{2,21} & \epsilon_{2,22} & \epsilon_{2,23} \\ \epsilon_{E_h,11} & \epsilon_{E_h,12} & \epsilon_{E_h,13} & \epsilon_{E_h,21} & \epsilon_{E_h,22} & \epsilon_{E_h,23} \\ \epsilon_{E_v,11} & \epsilon_{E_v,12} & \epsilon_{E_v,13} & \epsilon_{E_v,21} & \epsilon_{E_v,22} & \epsilon_{E_v,23} \end{array}\right) = \left(\begin{array}{ccc|ccc} -2 & 2 & 0 & 2 & 0 & -2 \\ 2 & 0 & -2 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right)$$

Adding half of the sum of the first row and the second row to the third row and to the fourth row, we obtain the matrix

$$\left(\begin{array}{ccc|ccc} -2 & 2 & 0 & 2 & 0 & -2 \\ 2 & 0 & -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The dimension of the solution space is $6-2=4$ since the rank of the matrix is 2. The four vectors below together form a basis of the solution space in $\mathbb{R}^6$.

$\mathbf{v}_a = {}^t(1,1,1 \mid 0,0,0)$, $\mathbf{v}_b = {}^t(0,0,0 \mid 1,1,1)$, $\mathbf{v}_c = {}^t(0,1,0 \mid 0,0,1)$, $\mathbf{v}_d = {}^t(0,0,1 \mid 0,1,0)$

In the rest, we determine all the Q-fundamental surfaces for the triangulation $T(2,1)$ of the $(2,1)$-lens space. Set $\mathbf{v} = a\mathbf{v}_a + b\mathbf{v}_b + c\mathbf{v}_c + d\mathbf{v}_d = {}^t(a,\ a+c,\ a+d \mid b,\ b+d,\ b+c)$ with $a,b,c,d \in \mathbb{R}$, the general solution of the matching equations. We consider when $\mathbf{v}$ represents a Q-fundamental surface.

Case of $a > 0$: Since the first entry is positive, the second and third entries are 0 by the square condition. Then we get $c = d = -a$, and fourth and the fifth entries are $b - a$. The square condition implies these entires are 0 (otherwise, both of them would be positive), and we have $b = a$ and $\mathbf{v} = {}^t(a, 0, 0 \mid a, 0, 0)$. $a \geq 1$ implies $\mathbf{v} = {}^t(a, 0, 0 \mid a, 0, 0) \geq {}^t(1, 0, 0 \mid 1, 0, 0)$. If $\mathbf{v}$ is Q-fundamental, then $\mathbf{v} = {}^t(1, 0, 0 \mid 1, 0, 0)$. Let $\mathbf{f}_1$ denote this vector, which is a candidate of a Q-fundamental solution.

Case of $a = 0$: In this case, the second, the third and the fourth entries are d, c and b respectively. Hence b, c, d are non-negative integers. Thus, if $\mathbf{v} = b\mathbf{v}_b + c\mathbf{v}_c + d\mathbf{v}_d$ represents a Q-fundamental solution, then $(b, c, d) = (1, 0, 0), (0, 1, 0)$ or $(0, 0, 1)$, and $\mathbf{v} = \mathbf{v}_b, \mathbf{v}_c$ or $\mathbf{v}_d$. $\mathbf{v}_b$ violates the square condition. We set $\mathbf{f}_2 = \mathbf{v}_c = (0, 1, 0 \mid 0, 0, 1)$, $\mathbf{f}_3 = \mathbf{v}_d = (0, 0, 1 \mid 0, 1, 0)$ as in Theorem 1.2 (i). Note that π-rotation about the axis E_v carries $\mathbf{f}_3$ to $\mathbf{f}_2$.

Actually, the three vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ are all Q-fundamental solutions. To see this, for any vector $\mathbf{x}$, let size($\mathbf{x}$) be the sum of all the elements of $\mathbf{x}$. Since size($\mathbf{f}_1$) $= 2$, size($\mathbf{f}_2$) $= 2$ and size($\mathbf{f}_3$) $= 2$, each of the three vectors can't be presented as a non-trivial linear combination of other candidates with non-negative integer coefficients.

$\mathbf{f}_1$ represents a Heegaard splitting torus which surrounds the core circle E_v. $\mathbf{f}_2$ and $\mathbf{f}_3$ represent projective planes. $2\mathbf{f}_2$ and $2\mathbf{f}_3$ represent the inessential 2-spheres surrounding the edges e_2 and e_1 respectively.

4. $(p, 1)$-lens spaces

In this section, we prove Theorem 1.2 (2), that is, we determine all the Q-fundamental surfaces for the triangulation $T(p, 1)$ of the $(p, 1)$-lens space with $p \geq 3$.

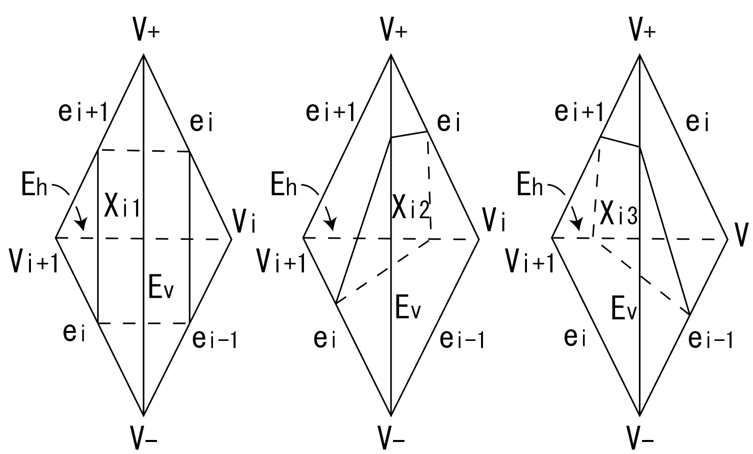

FIGURE 8

For $T(p,1)$, the senses of X_{i1}, X_{i2}, X_{i3} with respect to an edge e are as below.

$$\epsilon_{e,i1} = \begin{cases} 1 \text{ if } e = e_{i-1} \\ -2 \text{ if } e = e_i \\ 1 \text{ if } e = e_{i+1} \\ 0 \text{ if } e = E_h \\ 0 \text{ if } e = E_v \\ 0 \text{ otherwise} \end{cases} \quad \epsilon_{e,i2} = \begin{cases} 0 \text{ if } e = e_{i-1} \\ 2 \text{ if } e = e_i \\ 0 \text{ if } e = e_{i+1} \\ -1 \text{ if } e = E_h \\ -1 \text{ if } e = E_v \\ 0 \text{ otherwise} \end{cases} \quad \epsilon_{e,i3} = \begin{cases} -1 \text{ if } e = e_{i-1} \\ 0 \text{ if } e = e_i \\ -1 \text{ if } e = e_{i+1} \\ 1 \text{ if } e = E_h \\ 1 \text{ if } e = E_v \\ 0 \text{ otherwise} \end{cases}$$

where suffix numbers are considered modulo p. Note that $\epsilon_{i,i1} = -2$ since the tetrahedron τ_i has two copies of the edge e_i, and the disk of type $i1$ intersects each of the copies at a single point with negative sign. Then we obtain the coefficient matrix of the Q-matching equations for the $(p,1)$-lens space as below.

$$\left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
-2 & 2 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 1 & 0 & -1 \\
1 & 0 & -1 & -2 & 2 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & -2 & 2 & 0 & 1 & 0 & -1 & & & & & & \\
 & & & & & & & \ddots & & & \ddots & & & \ddots & & & & \\
 & & & & & & & & & 1 & 0 & -1 & -2 & 2 & 0 & 1 & 0 & -1 \\
1 & 0 & -1 & & & & & & & & & & 1 & 0 & -1 & -2 & 2 & 0 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & & \cdots & & 0 & -1 & 1 & 0 & -1 & 1 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & & \cdots & & 0 & -1 & 1 & 0 & -1 & 1
\end{array}\right)$$

Let r_i be the i-th row. Adding $(r_1+r_2+\cdots+r_p)/2$ to the $(p+1)$-st row and to the $(p+2)$-nd row, we can reduce the matrix as below.

$$\left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
-2 & 2 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 1 & 0 & -1 \\
1 & 0 & -1 & -2 & 2 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & -2 & 2 & 0 & 1 & 0 & -1 & & & & & & \\
 & & & & & & & \ddots & & & \ddots & & & \ddots & & & & \\
 & & & & & & & & & 1 & 0 & -1 & -2 & 2 & 0 & 1 & 0 & -1 \\
1 & 0 & -1 & & & & & & & & & & 1 & 0 & -1 & -2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

The rank of this matrix is p since the 2nd element of the i-th block (the $(3(i-1)+2)$-nd element) is non-zero only for the i-th row vector for $1 \leq i \leq p$. Hence the dimension of the solution space is $3p - p = 2p$. We obtain a basis $\mathbf{s}'_1, \mathbf{s}'_2, \cdots, \mathbf{s}'_p$, $\mathbf{t}'_1, \mathbf{t}'_2, \cdots, \mathbf{t}'_p$ of the solution space considered in $\mathbb{R}^{3p}$, where

$\mathbf{s}'_1 = {}^t(1,1,1 \mid 0,0,0 \mid 0,0,0 \mid \cdots \mid 0,0,0 \mid 0,0,0)$,
$\mathbf{s}'_2 = {}^t(0,0,0 \mid 1,1,1 \mid 0,0,0 \mid \cdots \mid 0,0,0 \mid 0,0,0)$,
$\vdots$
$\mathbf{s}'_p = {}^t(0,0,0 \mid 0,0,0 \mid 0,0,0 \mid \cdots \mid 0,0,0 \mid 1,1,1)$,
$\mathbf{t}'_1 = {}^t(0,0,2 \mid 0,1,0 \mid 0,0,0 \mid 0,0,0 \mid \cdots 0,0,0 \mid 0,0,0 \mid 0,1,0)$,

$\mathbf{t}'_2 = {}^t(0,1,0 \mid 0,0,2 \mid 0,1,0 \mid 0,0,0 \mid \cdots 0,0,0 \mid 0,0,0 \mid 0,0,0)$,
$\vdots$
$\mathbf{t}'_p = {}^t(0,1,0 \mid 0,0,0 \mid 0,0,0 \mid 0,0,0 \mid \cdots 0,0,0 \mid 0,1,0 \mid 0,0,2)$.
Note that $\mathbf{s}'_i$ and $\mathbf{t}'_i$ are obtained from $\mathbf{s}'_1$ and $\mathbf{t}'_1$ respectively by $(2\pi(i-1)/p)$-rotation about the axis E_v. Each $\mathbf{s}'_i$ is a solution of the Q-matching equations for any triangulation of any 3-manifold. (The sum of the three columns of the k-th block (the $(3(k-1)+1)$-st, the $(3(k-1)+2)$-nd and the $(3(k-1)+3)$-rd columns) is zero for $1 \le k \le p$.)

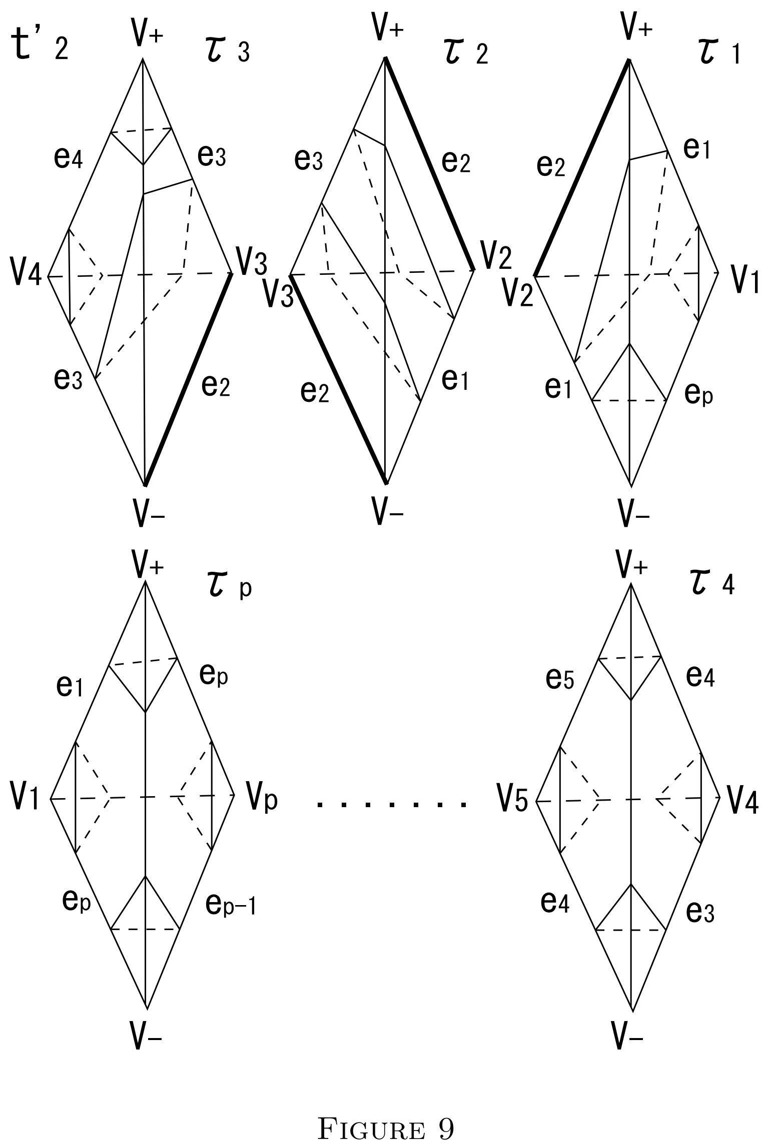

FIGURE 9

$\mathbf{t}'_i$ represents the normal surface which is the inessential 2-sphere surrounding the edge e_i (see Figure 9), and hence is a solution of the Q-matching equations. We shall see these vectors are linearly independent. We set the general solution
$\mathbf{v} = a_1\mathbf{s}'_1 + a_2\mathbf{s}'_2 + \cdots + a_p\mathbf{s}'_p + b_1\mathbf{t}'_1 + b_2\mathbf{t}'_2 + \cdots + b_p\mathbf{t}'_p =$

$= {}^t(a_1,\ a_1+b_2+b_p,\ a_1+2b_1 \mid \cdots \mid a_i,\ a_i+b_{i+1}+b_{i-1},\ a_i+2b_i \mid \cdots$
$\cdots \mid a_p,\ a_p+b_1+b_{p-1},\ a_p+2b_p)$

with $a_1,\cdots,a_p,b_1,\cdots,b_p \in \mathbb{R}$. If $\mathbf{v}=0$, considering the 1st element of the i-th block, we have $a_i=0$ for all i. Then $b_k=0$ follows from the $3k$ entry $(1 \le k \le p)$. Hence the $2p$ vectors are linearly independent.

We consider when the general solution $\mathbf{v}$ represents a Q-fundamental surface. Since $\mathbf{v}$ represents a normal surface, all the elements of $\mathbf{v}$ are non-negative integers. From the 1st element of the i-th block, a_i is a non-negative integer for $1 \le i \le p$. Then $b_i \in \mathbb{Z}/2$ because the 3rd element of the i-th block a_i+2b_i is an integer, where $\mathbb{Z}/2$ is the set of all the integers and the half integers.

The case of $b_i \ge 0$ for all i: We consider the case where $b_i \ge 0$ for $1 \le i \le p$. If a_i were positive, then both the 1st element of the i-th block a_i and the 3rd element of the i-th block a_i+2b_i would be positive, contradicting the square condition. Hence $a_i=0$ for all i. Then

$\mathbf{v} = b_1\mathbf{t}'_1 + b_2\mathbf{t}'_2 + \cdots + b_p\mathbf{t}'_p$
$= {}^t(0,\ b_2+b_p,\ 2b_1 \mid 0,\ b_3+b_1,\ 2b_2 \mid \cdots \mid 0,\ b_{i+1}+b_{i-1},\ 2b_i \mid \cdots \mid 0,\ b_1+b_{p-1},\ 2b_p) \cdots (*)$

(i) We consider the case where $b_i \ge 1$ for some i.

$\mathbf{v} \ge \mathbf{t}'_i = {}^t(0,0,0 \mid \cdots \mid 0,0,0 \mid 0,1,0 \mid 0,0,2 \mid 0,1,0 \mid 0,0,0 \mid \cdots \mid 0,0,0)$ by $(*)$, where the i-th block is $(0,0,2)$. When $\mathbf{v}$ represents a Q-fundamental surface, $\mathbf{v}=\mathbf{t}'_i$. Thus we obtained a candidate of a Q-fundamental solution.

(ii) The case where $b_i < 1$ for all i.

We have $b_k = 0$ or $1/2$ for each k since $b_k \in \mathbb{Z}/2$.

If $b_j = 1/2$, then $b_{j+1}=0$. Otherwise, $b_j=1/2$ and $b_{j+1}=1/2$ for some j, and both the 2nd element of the j-th block $b_{j+1}+b_{j-1}$ and the 3rd element of the j-th block $2b_j$ would be positive, contradicting the square condition.

If $b_k = 1/2$ for some k, then $b_{k+2}=1/2$. Because the 2nd element $b_{k+2}+b_k$ of the $(k+1)$-st block is an integer.

Hence p is even and $b_{\text{odd}}=1/2$ and $b_{\text{even}}=0$ or vice versa ($b_{\text{odd}}=0$ and $b_{\text{even}}=1/2$). Thus we obtained candidates of Q-fundamental solutions

$\mathbf{f}'_2 = (1/2)\mathbf{t}'_1 + (1/2)\mathbf{t}'_3 + \cdots + (1/2)\mathbf{t}'_{p-1} = {}^t(0,0,1 \mid 0,1,0 \mid 0,0,1 \mid 0,1,0 \mid \cdots \mid 0,0,1 \mid 0,1,0)$

and

$\mathbf{f}'_3 = (1/2)\mathbf{t}'_2 + (1/2)\mathbf{t}'_4 + \cdots + (1/2)\mathbf{t}'_p = {}^t(0,1,0 \mid 0,0,1 \mid 0,1,0 \mid 0,0,1 \mid \cdots \mid 0,1,0 \mid 0,0,1)$.

The case where $b_i < 0$ for some i:

First, we establish the next lemma.

Lemma 4.1. *If $b_i < 0$ for some i, then $a_i + 2b_i = 0$, $a_i + b_{i+1} + b_{i-1} = 0$ and $b_1 = b_2 = b_3 = \cdots = b_p$.*

Proof. Since b_i is negative, and since the 3rd element of the i-th block a_i+2b_i is non-negative, $a_i \geq -2b_i > 0$. Then the 1st element of the i-th block a_i is positive, and the square condition requires the other two elements of the block are zero, i.e., $a_i+2b_i = 0$ and $a_i+b_{i+1}+b_{i-1} = 0$. By subtracting the former equation from the latter, we obtain $b_{i-1} - b_i = b_i - b_{i+1}$. Hence either $b_{i-1} < b_i < b_{i+1}$, $b_{i-1} = b_i = b_{i+1}$ or $b_{i-1} > b_i > b_{i+1}$. Then $b_{i-1} \geq b_i \geq b_{i+1}$ or $b_{i-1} \leq b_i \leq b_{i+1}$ holds. We consider the case of $b_{i-1} \geq b_i \geq b_{i+1}$. (Similar argument will do for the case of $b_{i-1} \leq b_i \leq b_{i+1}$. We omit it.) We will prove $0 > b_i \geq b_{i+1} \geq \cdots \geq b_{i+n}$ for any positive integer n by induction. Suppose that $0 > b_i \geq b_{i+1} \geq \cdots \geq b_{i+k} \quad \cdots$ (i). Since b_{i+k} is negative we have $b_{i+k-1} < b_{i+k} < b_{i+k+1}$, $b_{i+k-1} = b_{i+k} = b_{i+k+1}$ or $b_{i+k-1} > b_{i+k} > b_{i+k+1}$ by a similar argument as the beginning of this proof. Then (i) implies $b_{i+k-1} \geq b_{i+k} \geq b_{i+k+1}$. Hence $0 > b_i \geq b_{i+1} \geq \cdots \geq b_{i+n}$ for all $n \in \mathbb{N}$. Considering the case of $n = p$, where $b_i \geq b_{i+1} \geq \cdots \geq b_{i+p} = b_i$, we obtain $b_i = b_{i+1} = \cdots = b_{i+p-1}$. □

By the above lemma, $b_1 = b_2 = \cdots = b_p \cdots$ (i), and hence $b_j < 0$ for all j. Then $a_j + 2b_j = 0$ follows again by Lemma 4.1. Hence (i) implies $a_j = -2b_j = -2b_1$ for all j, and $a_1 = a_2 = \cdots = a_p$. Thus
$\mathbf{v} = {}^t(a_1, 0, 0 \mid a_1, 0, 0 \mid \cdots \mid a_1, 0, 0) \geq {}^t(1, 0, 0 \mid 1, 0, 0 \mid \cdots \mid 1, 0, 0) = \mathbf{f}'_1$.
Since $\mathbf{v}$ is Q-fundamental, $\mathbf{v} = \mathbf{f}'_1$, which is a candidate of a Q-fundamental solution.

Thus we have obtained all the candidates of Q-fundamental solutions $\mathbf{t}'_i, \mathbf{f}'_2, \mathbf{f}'_3, \mathbf{f}'_1$ satisfying the square condition. In fact, these four vectors are all Q-fundamental solutions.

Since $\mathrm{size}(\mathbf{t}'_i) = 4$, $\mathrm{size}(\mathbf{f}'_2) = p$, $\mathrm{size}(\mathbf{f}'_3) = p$ and $\mathrm{size}(\mathbf{f}'_1) = p$, and since $p \geq 3$, the minimal size of a non-trivial linear combination of these vectors with non-negative integer coefficients is $3 + 3 = 6$. Hence $\mathbf{t}'_i$ is Q-fundamental.

$\mathbf{f}'_1$ is Q-fundamental because the 1st element is non-zero and those of the other candidates are zero.

$\mathbf{f}'_2$ is Q-fundamental since the 3rd element of the 2nd block is 1 and those of the other candidates are 0 or 2.

$\mathbf{f}'_3$ is Q-fundamental because $(2\pi(i-1)/p)$-rotation about the axis E_v brings $\mathbf{f}'_3$ to $\mathbf{f}'_2$.

$\mathbf{t}'_i$ represents the inessential 2-sphere surrounding the edge e_i (Figure 9). $\mathbf{f}'_2$ and $\mathbf{f}'_3$ represent a non-orientable closed surface with maximum Euler characteristic, which is the connected sum of $p/2$ projective planes (p is even) (Figure 10). $\mathbf{f}'_1$ represents a Heegaard splitting torus which surrounding the core circle E_v (Figure 11).

5. A basis of the solution space of the Q-matching equations for $T(p, q)$

In this section, we prove Lemma 1.3.

For $T(p, q)$, the senses of X_{i1}, X_{i2}, X_{i3} with respect to an edge e are as below. See Figure 4.

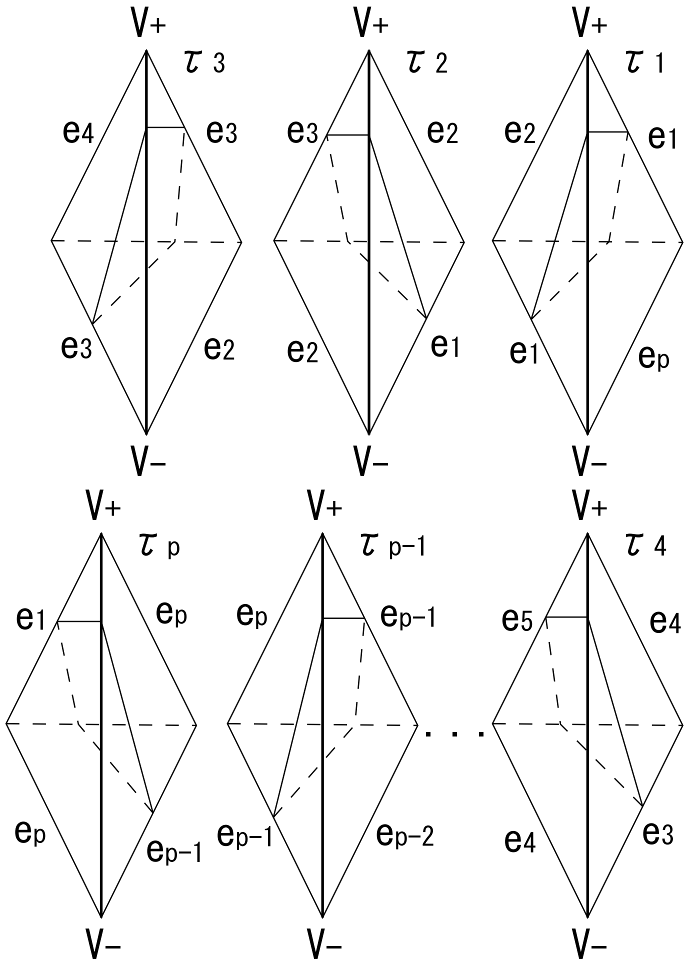

FIGURE 10

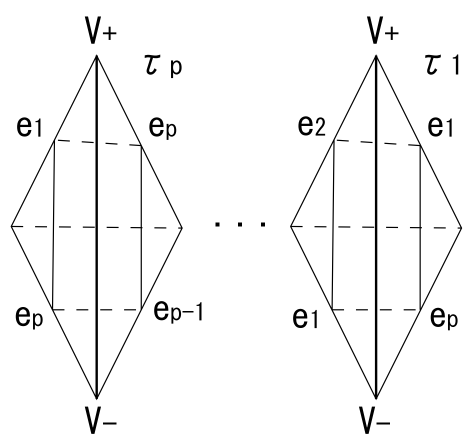

FIGURE 11

$$\epsilon_{e,i1} = \begin{cases} 1 \text{ if } e = e_{i-q} \\ -1 \text{ if } e = e_{i-(q-1)} \\ -1 \text{ if } e = e_i \\ 1 \text{ if } e = e_{i+1} \\ 0 \text{ if } e = E_h \\ 0 \text{ if } e = E_v \\ 0 \text{ otherwise} \end{cases} \quad \epsilon_{e,i2} = \begin{cases} 0 \text{ if } e = e_{i-q} \\ 1 \text{ if } e = e_{i-(q-1)} \\ 1 \text{ if } e = e_i \\ 0 \text{ if } e = e_{i+1} \\ -1 \text{ if } e = E_h \\ -1 \text{ if } e = E_v \\ 0 \text{ otherwise} \end{cases} \quad \epsilon_{e,i3} = \begin{cases} -1 \text{ if } e = e_{i-q} \\ 0 \text{ if } e = e_{i-(q-1)} \\ 0 \text{ if } e = e_i \\ -1 \text{ if } e = e_{i+1} \\ 1 \text{ if } e = E_h \\ 1 \text{ if } e = E_v \\ 0 \text{ otherwise} \end{cases}$$

where suffix numbers are considered modulo p. The coefficient matrix of the Q-matching equations is a $(p+2)\times 3p$ matrix. For the i-th row $(1 \leq i \leq p)$, the i-th and the $(q+(i-1))$-st blocks are $(-1,1,0)$, the $(i-1)$-st and the $(q+i)$-th blocks are $(1,0,-1)$ and the other blocks are $(0,0,0)$. All the blocks of the $(p+1)$-st and the $(p+2)$-nd rows are $(0,-1,1)$.

Let r_i be the i-th row. By adding $(r_1+r_2+\cdots+r_p)/2$ to the $(p+1)$-st row and to the $(p+2)$-nd row, we can deform these rows to zero vectors. Hence the rank is smaller than or equal to p. We set

$\mathbf{s}_1 = {}^t(1,1,1 \mid 0,0,0 \mid \cdots \mid 0,0,0)$,

$\mathbf{t}_1 = {}^t(0,0,1 \mid 0,0,0 \mid \cdots \mid 0,0,0 \mid 0,0,1 \mid 0,1,0 \mid 0,0,0 \mid \cdots \mid 0,1,0)$.

The 1st and the q-th blocks of $\mathbf{t}_1$ are $(0,0,1)$ and the $(q+1)$-st and the p-th blocks $(0,1,0)$. The other blocks of $\mathbf{t}_1$ are $(0,0,0)$. As we see below, these are solutions of the Q-matching equations. $\mathbf{s}_i$ and $\mathbf{t}_i$ are obtained from $\mathbf{s}_1$ and $\mathbf{t}_1$ respectively by $(2\pi(i-1)/p)$-rotation about the axis E_v. The i-th and the $(i+(q-1))$-st blocks of $\mathbf{t}_i$ are $(0,0,1)$ and the $(i-1)$-st and the $(i+q)$-th blocks $(0,1,0)$. $\mathbf{s}_i$ is a solution of the Q-matching equations for any triangulation of any 3-manifold. However, it does not satisfy the square condition. $\mathbf{t}_i$ represents the normal surface which is the inessential 2-sphere surrounding the edge e_i and hence is a solution of the Q-matching equations.

We prove that $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p$ are linearly independent. We solve the system of linear equations below.

$$\begin{aligned} \mathbf{0} &= a_1\mathbf{s}_1 + a_2\mathbf{s}_2 + \cdots + a_p\mathbf{s}_p + b_1\mathbf{t}_1 + b_2\mathbf{t}_2 + \cdots + b_p\mathbf{t}_p \\ &= {}^t(a_1,\ a_1+b_2+b_{p-q+1},\ a_1+b_1+b_{p-q+2} \mid a_2,\ a_2+b_3+b_{p-q+2},\ a_2+b_2+b_{p-q+3} \mid \cdots \\ &\cdots \mid a_i,\ a_i+b_{i+1}+b_{p-q+i},\ a_i+b_i+b_{p-q+i+1} \mid \cdots \mid a_p,\ a_p+b_1+b_{p-q},\ a_p+b_p+b_{p-q+1}\ ). \end{aligned}$$

We have $a_i = 0$ from the 1st element of the i-th block $(1 \leq i \leq p)$. Hence from the 2nd and the 3rd elements of the i-th block $b_{i+1}+b_{p-q+i} = 0 \ \cdots(1)$ and $b_i + b_{p-q+i+1} = 0 \ \cdots(2)$. Since this holds for all i and since we consider suffix numbers modulo p, we obtain the equation (3) below. (Substituting $j+q$ for i in (1), we have $b_{p+j} = -b_{j+q+1}$. Then the first equation of (3) follows. For the second equation of (3), we replace i with $j+q+1$ in (2). Then $-b_{j+q+1} = b_{p+j+(q+1)-(q-1)}$ holds.)

$$b_j = -b_{j+(q+1)} = b_{j+(q+1)-(q-1)} = b_{j+2} \ \cdots(3)$$

(i) The case of $q = 2k+1$ $(k \in \mathbb{N})$.

We get $b_i = b_{i+2} = b_{i+4} = \cdots = b_{i+2k} = b_{i+q-1} \cdots (4)$. From (2), $b_i = -b_{i+q-1}$ follows. This together with (4) implies $b_i = -b_i$, that is, $b_i = 0$ for all i. Hence the vectors are linearly independent.

(ii) The case of $q = 2m$ ($m \in \mathbb{N}$).

$b_i = -b_{i+(q+1)}$ follows from (1). On the other hand, $-b_{i+(q+1)} = -b_{i+2m+1} = -b_{i+2(m-1)+1} = -b_{i+2(m-2)+1} = \cdots = -b_{i+1}$ by (3). Hence we get $b_i = -b_{i+1} \cdots (5)$.

Note that p is odd since p and q are coprime. We set $p = 2n + 1$ ($n \in \mathbb{N}$). Since we consider indices modulo p, $b_i = b_{i+p}$ holds. (3) implies $b_{i+p} = b_{i+2n+1} = b_{i+2n-1} = b_{i+2n-3} = \cdots = b_{i+1}$. Hence $b_i = b_{i+1} \cdots (6)$.

By (5) and (6), $b_i = 0$ for all i.

Thus we have shown that the vectors $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p$ are linearly independent by (i) and (ii).

Similarly, we can see that the first p row vectors of the coefficient matrix of the Q-matching equations are linearly independent. Then the dimension of the solution space is $3p - p = 2p$. Hence $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p$ form a basis of the solution space.

6. $(p, 2)$-LENS SPACES

In this section, we prove Theorem 1.2 (3), that is, we determine all the Q-fundamental surfaces for the triangulation $T(p, 2)$ of the $(p, 2)$-lens space with $p \geq 5$. In Section 4, we have obtained a basis of the solution space ($\subset \mathbb{R}^{3t}$) of the Q-matching equations for the (p, q)-lens space. A linear combination of them gives a general solution. For $T(p, 2)$, it is

$$\begin{aligned}\mathbf{v} &= a_1\mathbf{s}_1 + b_1\mathbf{t}_1 + a_2\mathbf{s}_2 + b_2\mathbf{t}_2 + \cdots + a_p\mathbf{s}_p + b_p\mathbf{t}_p \\ &= {}^t(a_1,\ a_1 + b_2 + b_{p-1},\ a_1 + b_1 + b_p \mid a_2,\ a_2 + b_3 + b_p,\ a_2 + b_2 + b_1 \mid \cdots \\ &\cdots \mid a_i,\ a_i + b_{i+1} + b_{i-2},\ a_i + b_i + b_{i-1} \mid \cdots \mid a_p,\ a_p + b_1 + b_{p-2},\ a_p + b_p + b_{p-1}\ ).\end{aligned}$$

where $a_i, b_i \in \mathbb{R}$. We consider when $\mathbf{v}$ represents a Q-fundamental surface.

The case where $a_j = 0$ for all j:

(i) The case where the 3rd element of the $(i + 1)$-st block $b_{i+1} + b_i > 0 \ \cdots (1)$ for some i.

In this case,

$$\begin{aligned}\mathbf{v} = {}^t(&0,\ b_2 + b_{p-1},\ b_1 + b_p \mid \cdots \mid 0,\ b_i + b_{i-3},\ b_{i-1} + b_{i-2} \mid 0,\ b_{i+1} + b_{i-2},\ b_i + b_{i-1} \mid \\ &0,\ b_{i+2} + b_{i-1},\ b_{i+1} + b_i \mid 0,\ b_{i+3} + b_i,\ b_{i+2} + b_{i+1} \mid \cdots \mid 0,\ b_1 + b_{p-2},\ b_p + b_{p-1}).\end{aligned}$$

The 2nd element of the $(i + 1)$-st block $b_{i+2} + b_{i-1} = 0 \ \cdots (2)$ by the square condition. Adding (2) to (1), we obtain $b_i + b_{i-1} + b_{i+2} + b_{i+1} > 0$. Hence either $b_i + b_{i-1} > 0$ or $b_{i+2} + b_{i+1} > 0$. We consider the former case. Ultimately we will show that $\mathbf{v} = \mathbf{t}_i$. (In the latter case, similar argument shows $\mathbf{v} = \mathbf{t}_{i+1}$, and we omit it.) Then $b_i + b_{i-1} > 0 \ \cdots (3)$, which is the 3rd element of the i-th block. Hence the 2nd element $b_{i+1} + b_{i-2} = 0 \ \cdots (4)$ by the square conditon. We get $b_{i-1} + b_{i-2} + b_{i+2} + b_{i+1} = 0 \ \cdots$ (5) from (2) and (4). The left hand side of (5) is the sum of the 3rd element of the $(i-1)$-st block $b_{i-1} + b_{i-2}$ and the 3rd element of the $(i+2)$-nd block $b_{i+2}+b_{i+1}$, which are both non-negative. Hence they are equal

to 0, i.e., $b_{i-1}+b_{i-2}=0\ \cdots(6)$ and $b_{i+2}+b_{i+1}=0\ \cdots(7)$. We assume, for a contradiction, that the 2nd element of the $(i-1)$-st block $b_i+b_{i-3}=0$. This together with (6) gives $b_{i-2}+b_{i-3}+b_i+b_{i-1}=0$, which is the sum of the 3rd element of the $(i-2)$-nd block and the 3rd element of the i-th block. Then we obtain $b_i+b_{i-1}=0$, which contradicts (3). Hence we get $b_i+b_{i-3}>0\ \cdots\ (8)$. Similar argument shows $b_{i+3}+b_i>0\ \cdots\ (9)$. (We assume the 2nd element of the $(i+2)$-nd block $b_{i+3}+b_i=0$, to obtain $b_{i+1}+b_i+b_{i+3}+b_{i+2}=0$ by (7). Considering the 3rd element of the $(i+1)$-th block and the 3rd element of the $(i+3)$-rd block, we can see $b_{i+1}+b_i=0$, contradicting (1).)
Then, by (1), (3), (8) and (9),

$$\begin{aligned}\mathbf{v}={}^t(&0,\ b_2+b_{p-1},\ b_1+b_p\mid\cdots\mid 0,\ b_{i-1}+b_{i-4},\ b_{i-2}+b_{i-3}\mid 0,\ b_i+b_{i-3},\ 0\mid\\ &0,\ 0,\ b_i+b_{i-1}\mid 0,\ 0,\ b_{i+1}+b_i\mid 0,\ b_{i+3}+b_i,\ 0\mid 0,\ b_{i+4}+b_{i+1},\ b_{i+3}+b_{i+2}\mid\cdots\\ &\cdots\mid 0,\ b_1+b_{p-2},\ b_p+b_{p-1})\\ \geq(&0,\ 0,\ 0\mid\cdots\mid 0,\ 0,\ 0\mid 0,\ 1,\ 0\mid\\ &0,\ 0,\ 1\mid 0,\ 0,\ 1\mid 0,\ 1,\ 0\mid 0,\ 0,\ 0\mid\cdots\\ &\cdots\mid 0,\ 0,\ 0)=\mathbf{t}_i.\end{aligned}$$

Since $\mathbf{v}$ is Q-fundamental, $\mathbf{v}=\mathbf{t}_i$, which is a candidate of a Q-fundamental solution. We write this vector $\mathbf{t}''_i$ to clarify that it is for a $(p,2)$-lens space.

(ii) The case where $b_i+b_{i+1}=0$ for all i.
Considering this condition for $i=1,2,\cdots,p$, we get a system of linear equations. Then we solve it to have $b_i=0$ for all i, since $q=2$ implies p is odd. Thus $\mathbf{v}=\mathbf{0}$, a contradiction.

The case where $a_j>0$ for some j:
Since the 1st element a_j in the j-th block is positive, by the square condition, the 3rd element of the j-th block $a_j+b_{j-1}+b_j=0$, and hence $b_{j-1}+b_j<0$. Then

$$\begin{aligned}\mathbf{v}={}^t(&a_1,\ a_1+b_2+b_{p-1},\ a_1+b_1+b_p\mid a_2,\ a_2+b_3+b_p,\ a_2+b_2+b_1\mid\cdots\\ &\cdots\mid a_{j-1},\ a_{j-1}+b_j+b_{j-3},\ a_{j-1}+b_{j-1}+b_{j-2}\mid a_j,\ 0,\ 0\mid\\ &a_{j+1},\ a_{j+1}+b_{j+2}+b_{j-1},\ a_{j+1}+b_{j+1}+b_j\mid\cdots\mid a_p,\ a_p+b_1+b_{p-2},\ a_p+b_p+b_{p-1}).\end{aligned}$$

Lemma 6.1. *If $b_{i-1}+b_i<0$ for some i, then $a_i=-(b_{i-1}+b_i)>0$ and either $b_{i-2}+b_{i-1}=b_{i-1}+b_i=b_i+b_{i+1}$, $b_{i-2}+b_{i-1}<b_{i-1}+b_i<b_i+b_{i+1}$ or $b_{i-2}+b_{i-1}>b_{i-1}+b_i>b_i+b_{i+1}$.*

Proof. Since $b_{i-1}+b_i$ is negative, and since the 3rd element of the i-th block $a_i+b_{i-1}+b_i\geq 0$, we have $a_i>a_i+b_{i-1}+b_i\geq 0$. In the i-th block, since the 1st element $a_i>0$, the 2nd and the 3rd elements are zero by the square condition. Summing up these elements $a_i+b_{i-1}+b_i=0$ and $a_i+b_{i-2}+b_{i+1}=0$, we get $2a_i+b_{i-2}+b_{i-1}+b_i+b_{i+1}=0$. Hence $-2a_i=b_{i-2}+b_{i-1}+b_i+b_{i+1}$, and we have either $b_{i-2}+b_{i-1}=-a_i=b_i+b_{i+1}$, $b_{i-2}+b_{i-1}<-a_i<b_i+b_{i+1}$ or $b_{i-2}+b_{i-1}>-a_i>b_i+b_{i+1}$.

In addition, we obtain $-a_i=b_{i-1}+b_i<0$ from the 3rd element. □

Lemma 6.2. *If $b_{l-1}+b_l<0$ for some l, then $-a_1=-a_2=\cdots=-a_p=b_1+b_2=b_2+b_3=\cdots=b_p+b_1$.*

Proof. Setting $i=l$ in Lemma 6.1, we have either $b_{l-2}+b_{l-1}=b_{l-1}+b_l=b_l+b_{l+1}$, $b_{l-2}+b_{l-1}<b_{l-1}+b_l<b_l+b_{l+1}$ or $b_{l-2}+b_{l-1}>b_{l-1}+b_l>b_l+b_{l+1}$. Then $b_{l-2}+b_{l-1}\leq b_{l-1}+b_l\leq b_l+b_{l+1}$ or $b_{l-2}+b_{l-1}\geq b_{l-1}+b_l\geq b_l+b_{l+1}$ holds. We consider the case of $b_{l-2}+b_{l-1}\leq b_{l-1}+b_l\leq b_l+b_{l+1}$. (Similar argument will do for the other case, and we omit it.) We will prove $0>b_{l-1}+b_l\geq b_{l-2}+b_{l-1}\geq\cdots\geq b_{l-1-n}+b_{l-n}$ for any positive integer n by induction. Suppose $0>b_{l-1}+b_l\geq b_{l-2}+b_{l-1}\geq\cdots\geq b_{l-1-k}+b_{l-k}$. Setting $i=l-k$ in Lemma 6.1, we get either $b_{l-2-k}+b_{l-1-k}=b_{l-1-k}+b_{l-k}=b_{l-k}+b_{l-k+1}$, $b_{l-2-k}+b_{l-1-k}>b_{l-1-k}+b_{l-k}>b_{l-k}+b_{l-k+1}$ or $b_{l-2-k}+b_{l-1-k}<b_{l-1-k}+b_{l-k}<b_{l-k}+b_{l-k+1}$. The second one contradicts the assumption of induction. Then we have $0>b_{l-1-k}+b_{l-k}\geq b_{l-2-k}+b_{l-1-k}$. Hence $0>b_{l-1}+b_l\geq\cdots\geq b_{l-1-n}+b_{l-n}$ for all $n\in\mathbb{N}$ by induction. We set $n=p$, to obtain $0>b_{l-1}+b_l\geq\cdots\geq b_{l-1-p}+b_{l-p}=b_{l-1}+b_l$, where we consider indices modulo p. Hence $0>b_{l-1}+b_l=\cdots=b_{l-1-p}+b_{l-p}$ follows.

By Lemma 6.1, $b_{l-1-m}+b_{l-m}=-a_{l-m}$ holds for $0\leq m\leq p$. □

By Lemma 6.2, $a_i=-(b_{i-1}+b_i)>0$ for $\forall i\in\{1,2,\cdots,p\}$. Hence, the 2nd and the 3rd elements are zero in each block by the square condition, and
$\mathbf{v}={}^t(a_1,\ 0,\ 0,\ |\cdots|\ a_j,\ 0,\ 0\ |\cdots|\ a_p,\ 0,\ 0)\geq{}^t(1,\ 0,\ ,0\ |\cdots|\ 1,\ 0,\ 0\ |\cdots|\ 1,\ 0,\ 0)$
Let $\mathbf{f}''_1$ denote the last vector. Since $\mathbf{v}$ is Q-fundamental, $\mathbf{v}=\mathbf{f}''_1$, which is a candidate of a Q-fundamental solution.

Thus we have obtained all the candidates of Q-fundamental solutions $\mathbf{t}''_1,\cdots,\mathbf{t}''_p$ and $\mathbf{f}''_1$ satisfying the square condition. In fact, $\mathbf{t}''_i$ and $\mathbf{f}''_1$ are Q-fundamental solutions. $\mathbf{f}''_1$ is Q-fundamental because the 1st elements of the blocks of $\mathbf{f}''_1$ are 1 and those of the other vectors $\mathbf{t}''_1,\cdots,\mathbf{t}''_p$ are 0. Since $\mathrm{size}(\mathbf{t}''_i)=4$, $\mathrm{size}(\mathbf{f}''_1)=p$ and $p\geq 5$, $\mathbf{t}''_i$ is a Q-fundamental solution.

$\mathbf{t}''_i$ represents the inessential 2-sphere surrounding the edge e_i. See Figure 12. $\mathbf{f}''_1$ represents a Heegaard splitting torus which surrounds the core circle E_v.

7. Q-fundamental surfaces in the (p,q)-lens space

In this section, we prove Lemma 1.4 and Theorems 1.5, 1.6. In general (p,q)-lens spaces, we consider Q-fundamental surfaces with no quadrilateral normal disks disjoint from the core circles E_v and E_h. In another words, we consider when $\mathbf{v}=a_1\mathbf{s}_1+\cdots+a_p\mathbf{s}_p+b_1\mathbf{t}_1+\cdots+b_p\mathbf{t}_p$ with $a_1=\cdots=a_p=0$ represents a Q-fundamental surface. We will obtain a restriction on b_i's.

Proof. We prove Lemma 1.4.

The j-th block of $\mathbf{v}$ is ${}^t(a_j,\ a_j+b_{j+1}+b_{p-q+j},\ a_j+b_j+b_{p-q+j+1})$ for $1\leq j\leq p$.

Since $\mathbf{v}\in\mathbb{Z}^{3p}$, we can see from the 1st element of each block that $a_k\in\mathbb{Z}$ for all $k\cdots$ (1).

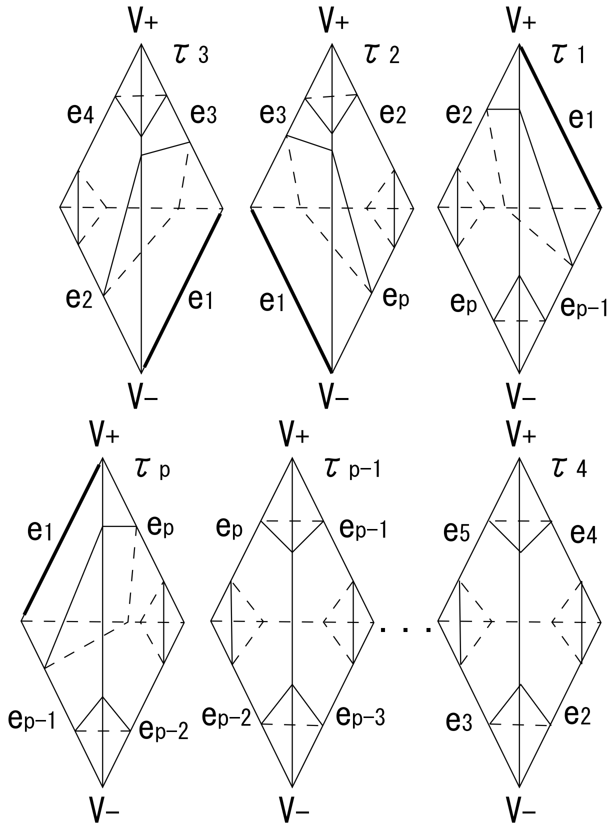

FIGURE 12

Let i be an arbitrary integer with $1 \leq i \leq p$. From the 3rd element of the i-th block, we can see $a_i + b_i + b_{p-q+i+1} \in \mathbb{Z}$. Then (1) implies $b_i + b_{p-q+i+1} \in \mathbb{Z} \cdots (2)$. On the other hand, we have $b_{i+2} + b_{p-q+i+1} \in \mathbb{Z} \cdots$ (3) from the 2nd element of the $(i+1)$-st block. By subtract (3) from (2), we obtain $\mathbb{Z} \ni (b_i + b_{p-q+i+1}) - (b_{i+2} + b_{p-q+i+1}) = b_i - b_{i+2} \cdots (4)$ for all i.

When p is odd, $b_i - b_{i+2},\ b_{i+2} - b_{i+4},\ \cdots,\ b_{i+2(p-2)} - b_{i+2(p-1)} \in \mathbb{Z}$ by (4), and hence $\mathcal{B} \subset \mathbb{Z} + b_1$, that is, the decimal places of all the b_i's coincide. In particular $b_i - b_{p-q+i+1} \in \mathbb{Z}$. This together with (2) implies $\mathbb{Z} \ni (b_i + b_{p-q+i+1}) + (b_i - b_{p-q+i+1}) = 2b_i \cdots (5)$. Hence $b_i \in \mathbb{Z}$ or $b_i \in \mathbb{Z} + 1/2$.

When p is even, $\mathcal{B}_0 \in \mathbb{Z} + b_0$ and $\mathcal{B}_1 \in \mathbb{Z} + b_1$ by (4). Since q is odd, $p - q + 1$ is even, and hence $b_i - b_{p-q+i+1} \in \mathbb{Z}$ for all i. Then we obtain (5) again, and, for $i = 0$ and 1, $\mathcal{B}_i \subset \mathbb{Z}$ or $\mathcal{B}_i \subset \mathbb{Z} + 1/2$. □

In the rest of this section we consider the case where $a_k = 0$ for all k. Then $\mathbf{v} = b_1\mathbf{t}_1 + \cdots + b_p\mathbf{t}_p$, $(b_1, \cdots b_p \in \mathbb{R})$. The i-th block of $\mathbf{v}$ is ${}^t(0,\ b_{i+1} + b_{p-q+i},\ b_j + b_{p-q+i+1})$ for $1 \leq i \leq p$. Let $\tilde{\mathbf{v}}$ be the vector obtained from $\mathbf{v}$ by deleting the 1st elements of all the blocks. The 1st and the 2nd elements of the i-th block of $\tilde{\mathbf{v}}$ are equal to the 2nd and the 3rd elements of the i-th block of $\mathbf{v}$.

Proof. We prove Theorem 1.5.

(I) Since $\mathbf{v}$ is Q-fundamental, $\tilde{\mathbf{v}}$ has at least one positive element $b_j + b_l$. Either both j and l are odd, or both even because p is even. We suppose they are even, i.e., $j, l \in 2\mathbb{Z} \cdots (*)$. (Similar argument will do for the case of $j, l \in 2\mathbb{Z}+1$. We omit it.) We set $\mathbf{v}' = 0\mathbf{t}_1 + b_2\mathbf{t}_2 + \cdots + 0\mathbf{t}_{p-1} + b_p\mathbf{t}_p$, which is obtained from $\mathbf{v} = b_1\mathbf{t}_1 + \cdots + b_p\mathbf{t}_p$ by replacing all the b_i's with i odd by 0. Then $\mathbf{v} \geq \mathbf{v}'$ and $\mathbf{v}' > \mathbf{0}$ by $(*)$. Since $\mathbf{v}$ is Q-fundamental, $\mathbf{v} = \mathbf{v}'$. Hence $\mathcal{B}_1 \subset \{0\}$. $\mathcal{B}_0 \subset \mathbb{Z}$ or $\mathcal{B}_0 \subset \mathbb{Z}+1/2$ by Lemma 1.4. This completes the proof of Theorem 1.5 (I).

(II) **The case where p is odd**. Set $b'_i = -1/2, 0$ or $1/2$ according as b_i is negative, 0 or positive. Let $\mathbf{v}''$ be a solution obtained from $\mathbf{v}$ by replacing each b_i with b'_i, that is, $\mathbf{v}'' = b'_1\mathbf{t}_1 + \cdots + b'_p\mathbf{t}_p$. For each element $b_i + b_j$ of $\tilde{\mathbf{v}}$, we will prove $b_i + b_j \geq b'_i + b'_j \geq 0$.

In case of $b_i, b_j > 0$, the condition $b_i, b_j \in \mathbb{Z}+1/2$ implies $b_i, b_j \geq 1/2$. Hence $b_i + b_j \geq 1/2 + 1/2 = b'_i + b'_j$ and $b'_i + b'_j = 1/2 + 1/2 = 1 > 0$.

We consider the case of $b_i < 0$ and $b_j > 0$ (similar for $b_i > 0$ and $b_j < 0$). Since $b_i + b_j$ is the element of $\mathbf{v}$, we have $b_i + b_j \geq 0$. Hence $b_i + b_j \geq 0 = -1/2 + 1/2 = b'_i + b'_j$ and $b'_i + b'_j = -1/2 + 1/2 = 0 \geq 0$.

The condition $b_i < 0$ and $b_j < 0$ contradicts that every element of $\mathbf{v}$ is non-negative.

Now we prove that there is an element $b_i + b_j$ of $\tilde{\mathbf{v}}$ with $b_i >$ and $b_j > 0$. Since $b_i, b_j \in \mathbb{Z}+1/2$, each of them is positive or negative. Suppose, for a contradiction, that for every element $b_i + b_j$ of $\tilde{\mathbf{v}}$ the signs of b_i and b_j are opposite. Then $sgn(b_i \times b_{i-q+1}) = -1 \cdots (1)$ and $sgn(b_{i+2} \times b_{i-q+1}) = -1$ from the 3rd element of the i-th block and the 2nd element of the $(i+1)$-st block respectively. Then we obtain $sgn(b_i \times b_{i-q+1}) \times sgn(b_{i+2} \times b_{i-q+1}) = 1$, that is, $sgn(b_i \times b_{i+2}) = 1 \cdots (2)$ for all i. Since p is odd, by applying (2) repeatedly $\operatorname{sgn}(b_1) = \operatorname{sgn}(b_3) = \cdots = \operatorname{sgn}(b_p) = \operatorname{sgn}(b_2) = \operatorname{sgn}(b_4) = \cdots = \operatorname{sgn}(b_{p-1})$. Thus the signs of all the elements of $\mathcal{B}$ coincide. In particular, $sgn(b_i \times b_{i-q+1}) = 1$, which contradicts (1). Hence there is an element $b_i + b_j$ of $\tilde{\mathbf{v}}$ with $b_i, b_j > 0$. Then $\mathbf{v}''$ has an element $b'_i + b'_j > 0$ since $sgn(b'_i) = sgn(b_i)$ and $sgn(b'_j) = sgn(b_j)$ by definition.

Thus $\mathbf{v} \geq \mathbf{v}'' > \mathbf{0}$. Since $\mathbf{v}''$ is a solution and $\mathbf{v}$ is Q-fundamental, $\mathbf{v} = \mathbf{v}''$.

The case where p is even. We can assume without loss of generality that $\mathcal{B}_0 = \{0\}$ and $\mathcal{B}_1 \subset \mathbb{Z}+1/2$. For each element $b_i + b_j$ of $\tilde{\mathbf{v}}$, i and j are both even or both odd.

Similar arguments as in the previous case will do except the followings.

In case of $b_i = 0$ and $b_j = 0$ (with i, j even), we obtain $b_i + b_j = 0 + 0 = b'_i + b'_j$. Hence $b_i + b_j \geq b'_i + b'_j \geq 0$ holds.

We prove that $\tilde{\mathbf{v}}$ has an element $b_k + b_l$ with $b_k, b_l > 0$. For all the elements $b_k + b_l$ with k, l odd, suppose that $b_k + b_l$ are of the opposite signs. For $\forall i \in \mathbb{Z}$ we have (1) and (2) again. Since q is odd, $-q+1$ is even, and by applying (2) repeatedly we obtain $sgn(b_i \times b_{i-q+1}) = 1$

for all odd integer i with $1 \le i \le p$. This contradicts (1). Thus the proof of Theorem 1.5 (II) is completed.

(III) Set $b_i^* = -1, 0$ or 1 according as b_i is negative, 0 or positive. Let $\mathbf{v}^*$ be the vector obtained from $\mathbf{v}$ by replacing each b_i with b_i^*.

The case where p is odd. For all the elements b_i+b_j of $\tilde{\mathbf{v}}$, we will prove $b_i+b_j \ge b_i^*+b_j^* \ge 0$ for all i, j.

If $b_i > 0$ and $b_j > 0$, then $b_i \ge 1$ and $b_j \ge 1$ because $b_i, b_j \in \mathbb{Z}$. Hence $b_i + b_j \ge 1 + 1 = b_i^* + b_j^* > 0$.

We consider the case of $b_i < 0$ and $b_j > 0$. (The same argument will do for the case of $b_i > 0$ and $b_j < 0$.) Since $b_i + b_j$ is an element of $\mathbf{v}$, it is non-negative. Hence we have $b_i + b_j \ge 0 = -1 + 1 = b_i^* + b_j^* \ge 0$.

When $b_i = 0$ and $b_j > 0$ (the proof is the same for the case of $b_i > 0$ and $b_j = 0$), $b_j \in \mathbb{Z}$ implies $b_j \ge 1$, and hence $b_i + b_j \ge 0 + 1 = b_i^* + b_j^* > 0$.

In case of $b_i = 0$ and $b_j = 0$, clearly $b_i + b_j \ge b_i^* + b_j^* \ge 0$ because $b_i + b_j = 0 + 0 = b_i^* + b_j^*$.

We prove that $\tilde{\mathbf{v}}$ has an element $b_i + b_j$ with ($b_i \ge 0$ and $b_j > 0$) or ($b_i > 0$ and $b_j \ge 0$). Suppose not. Then, for each element $b_i + b_j$ of $\tilde{\mathbf{v}}$, either (1) $b_i = b_j = 0$ or (2) b_i and b_j are of opposite signs.

Suppose $b_k = 0$ for some k. Then (1) holds rather than (2) for the 2nd element b_k+b_{k-q+1} of the k-th block of $\tilde{\mathbf{v}}$, that is, $b_{k-q+1} = 0$. This implies that the 1st element $b_{k+2} + b_{k-q+1}$ of the $(k + 1)$-st block of $\tilde{\mathbf{v}}$ satisfies (1). Then we have $b_{k+2} = 0$. Hence $b_k = 0$ implies $b_{k+2} = 0$ for all $k \cdots (3)$. Since p is odd, $b_k = 0$ for all k. Then $\tilde{\mathbf{v}} = \mathbf{0}$, and hence $\mathbf{v} = \mathbf{0}$, contradicting that it is Q-fundamental. Hence $b_k \ne 0$ for all k.

Thus we can assume that, for all the elements $b_i + b_j$ of $\tilde{\mathbf{v}}$, (2) holds, i.e., b_i and b_j are of opposite signs. Then we obtain a contradiction by similar arguments as in the former half of (II). Hence $\tilde{\mathbf{v}}$ has an element $b_i + b_j$ with ($b_i \ge 0$ and $b_j > 0$) or ($b_i > 0$ and $b_j \ge 0$).

Thus $\mathbf{v} \ge \mathbf{v}^* > 0$. Since $\mathbf{v}$ is a Q-fundamental, $\mathbf{v} = \mathbf{v}^*$.

The case where p is even. The proof is similar to the previous case, and to the latter half of (II). We omit it. This completes the proof of Theorem 1.5 (III). □

Proof. We prove Theorem 1.6. Suppose, for a contradiction, that $\mathbf{v}$ is Q-fundamental. Then $b_l = 1/2$ or $-1/2$ for all l by (II) in Theorem 1.5. Note that an element $b_i + b_j$ with $b_i = -1/2$ and $b_j = -1/2$ cannot exist. In the expression

$$\begin{aligned}\tilde{\mathbf{v}} = {}^t(&b_2 + b_{p-q+1},\ b_1 + b_{p-q+2} \mid b_3 + b_{p-q+2},\ b_2 + b_{p-q+3} \mid \cdots \\ &\cdots \mid b_{i+1} + b_{p-q+i},\ b_i + b_{p-q+i+1} \mid \cdots \mid b_1 + b_{p-q},\ b_p + b_{p-q+1}),\end{aligned}$$

each b_i appears four times. An element $b_i + b_j$ is equal to 0 if and only if precisely one of b_i and b_j is equal to $-1/2$. Thus the number of elements equal to 0 is (the number of b_i's equal to $-1/2$) $\times 4$. In particular it is even $\cdots (*)$.

The square condition implies that each block of $\tilde{\mathbf{v}}$ has one or two elements equal to 0. Let m be the number of blocks with their two elements equal to 0.

We will prove that the blocks with their two elements equal to 0 separate into pairs. Suppose that the block for τ_i has the two elements equal to 0. There are four patterns below.

(i) When $b_i = -1/2$ and $b_{i+1} = -1/2$, the pair of blocks for τ_i and τ_{i+q} are $\mathbf{0}$.
(ii) When $b_{i-q} = -1/2$ and $b_i = -1/2$, the pair of blocks for τ_i and τ_{i-1} are $\mathbf{0}$.
(iii) When $b_{i-q+1} = -1/2$ and $b_{i+1} = -1/2$, the pair of blocks for τ_i and τ_{i+1} are $\mathbf{0}$.
(iv) When $b_{i-q} = -1/2$ and $b_{i-q+1} = -1/2$, the pair of blocks for τ_i and τ_{i-q} are $\mathbf{0}$.

Note that the block for τ_i is $\begin{pmatrix} b_{i+1} + b_{i-q} \\ b_i + b_{i-q+1} \end{pmatrix}$, while the blocks for τ_{i-1}, τ_{i+1}, τ_{i+q} and τ_{i-q} are $\begin{pmatrix} b_i + b_{i-q-1} \\ b_{i-1} + b_{i-q} \end{pmatrix}$, $\begin{pmatrix} b_{i+2} + b_{i-q+1} \\ b_{i+1} + b_{i-q+2} \end{pmatrix}$, $\begin{pmatrix} b_{i+q+1} + b_i \\ b_{i+q} + b_{i+1} \end{pmatrix}$ and $\begin{pmatrix} b_{i-q+1} + b_{i-2q} \\ b_{i-q} + b_{i-2q+1} \end{pmatrix}$ respectively. Precisely one block has the same b_j's equal to $-1/2$ as that for τ_i.

Hence $m = 2k$ for some non-negative integer k, and the number of elements equal to 0 of $\tilde{\mathbf{v}}$ is $2m + 1 \times (p - m) = 2 \times 2k + 1 \times (p - 2k) = p + 2k$. Since p is odd, $p + 2k$ is odd. This contradicts $(*)$. □

8. Examples of fundamental surfaces

In this section, several examples of fundamental surfaces in lens spaces are given. The authors have comfirmed that they are actually Q-fundamental by computer except for the last one in the $(418, 153)$-lens space. By Lemma 1.7, they are fundamental with respect to Haken's matching equations except the inessential torus in the $(18, 7)$-lens space. We begin this section with showing the lemma.

Recall that a closed surface F in a lens space intersects each of E_v and E_h in odd number of points if and only if F is non-orientable.

Proof. We prove Lemma 1.7.

Suppose, for a contradiction, that F is decomposed as $F = F_1 + F_2$. Since F intersects E_v and E_h in a single point, one of F_1 or F_2, say F_1 intersects E_v and E_h in a single point, and F_2 is disjoint from $E_v \cup E_h$. Then F_2 is ${}^t(1, 0, 0 \mid 1, 0, 0 \mid \cdots \mid 1, 0, 0)$, the Heegaard splitting torus surrounding E_v and E_h, which is the only normal surface disjoint from $E_v \cup E_h$. Note that F_2 has a normal disks of type X_{k1} in each tetrahedron τ_k. By assumption, F has a normal disk of type X_{k2} or X_{k3} which cannot exist together with a normal disk of X_{k1}. This is a contradiction. □

We consider the (p, q)-lens space with p even and $q \geq 3$. Then $\mathbf{h} = (\sum_{k=1}^{p/2} \mathbf{t}_{2k-1})/2 = {}^t(0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 1\ 0| \cdots |0\ 0\ 1|0\ 1\ 0)$ is fundamental and not Q-fundamental. In fact, it is geometrically compressible. (For the definition, see section 2.)

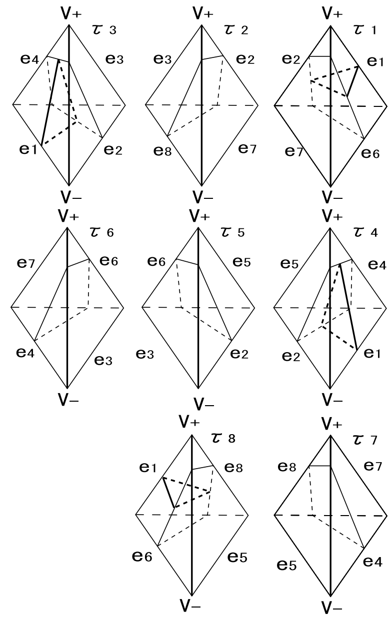

FIGURE 13

For example, in the (8, 3)-lens space, the normal surface $\mathbf{h} = {}^t(0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 1\ 0)$ is not Q-fundamental. It has a compressing disk as shown in Figure 13, and compressing yields the Q-fundamental surface $\mathbf{h} - \mathbf{t}_1 = {}^t(0\ 0\ 0|0\ 1\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 0\ 0)$ which represents a non-orientable closed surface with maximal Euler characteristic, a Klein bottle in this case. See Figure 14.

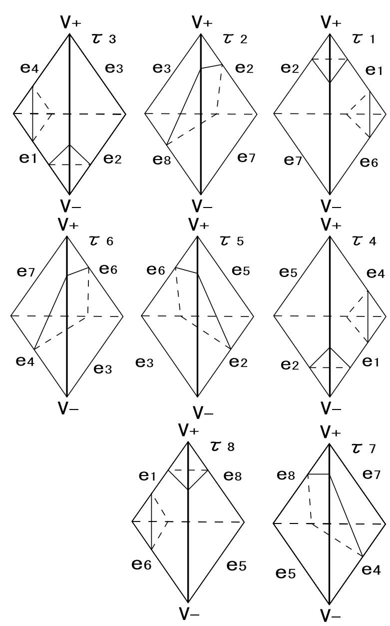

FIGURE 14

In case of $(16, 3)$-lens space, there are two ways of performing compressing operations twice. Both

$\mathbf{h} - \mathbf{t}_1 - \mathbf{t}_9$

$= {}^t(0\ 0\ 0|0\ 1\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 0\ 0|0\ 0\ 0|0\ 1\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 0\ 0)$

and

$\mathbf{h} - \mathbf{t}_1 - \mathbf{t}_7$

$= {}^t(0\ 0\ 0|0\ 1\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 1|0\ 0\ 0|0\ 0\ 0|0\ 1\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 1\ 0|0\ 0\ 1|0\ 0\ 0)$

are Q-fundamental surfaces which represent non-orientable closed surfaces with maximal Euler characteristics, In the $(16, 3)$-lens space, such a surface is the connected sum of four projective planes as shown in [2].

There is a Q-fundamental surface with some a_k non-zero. For example, $(18, 7)$-lens space has such one with the coordinate

${}^t(0\ 0\ 1|0\ 0\ 0|0\ 0\ 0|0\ 1\ 0|0\ 0\ 1|0\ 0\ 0|0\ 1\ 0|0\ 0\ 0|1\ 0\ 0|1\ 0\ 0|0\ 0\ 0|0\ 1\ 0|$
$0\ 0\ 0|0\ 0\ 1|0\ 1\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 1),$

which represents a compressible torus.

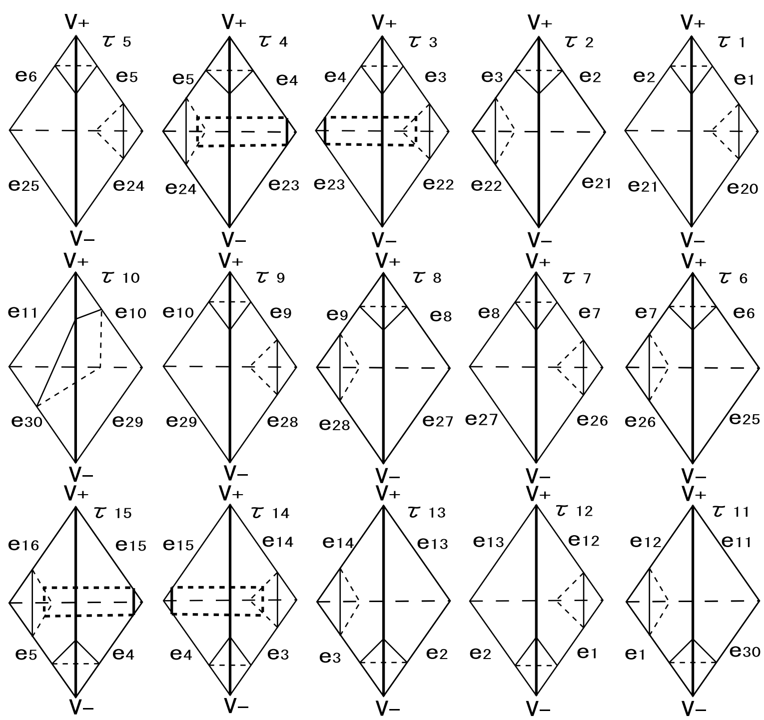

FIGURE 15

We consider the $(30, 11)$-lens space, where $\mathbf{h}$ has five compressing disks, and compression along them yields the normal surface

$\mathbf{h} - \mathbf{t}_1 - \mathbf{t}_3 - \mathbf{t}_5 - \mathbf{t}_7 - \mathbf{t}_9$

$= {}^t(0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 1|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|0\ 0\ 0|$

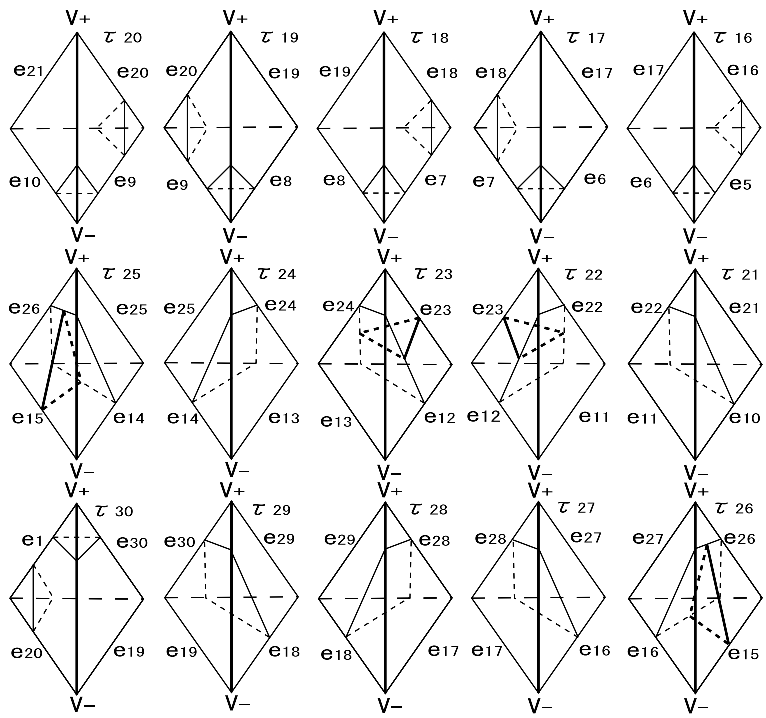

Figure 16

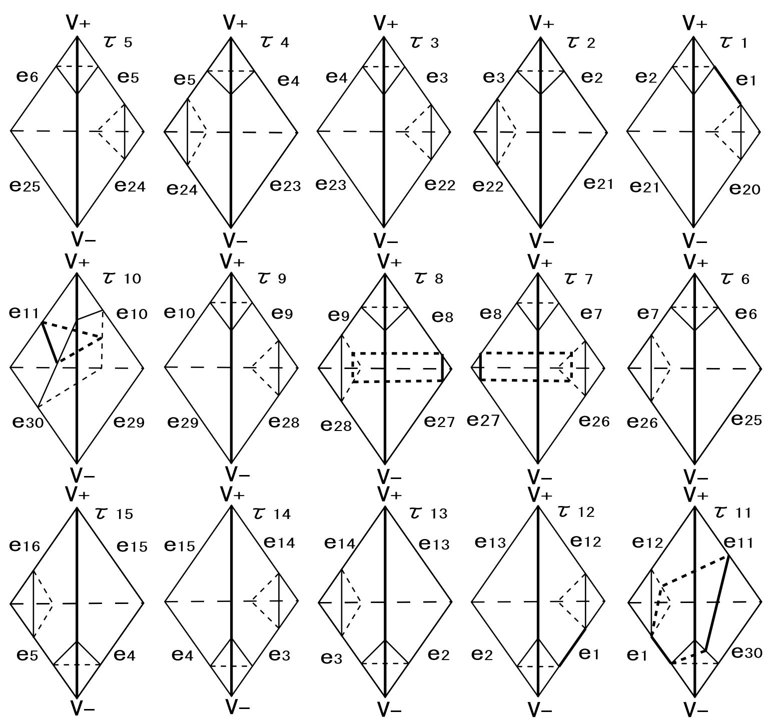

Figure 17

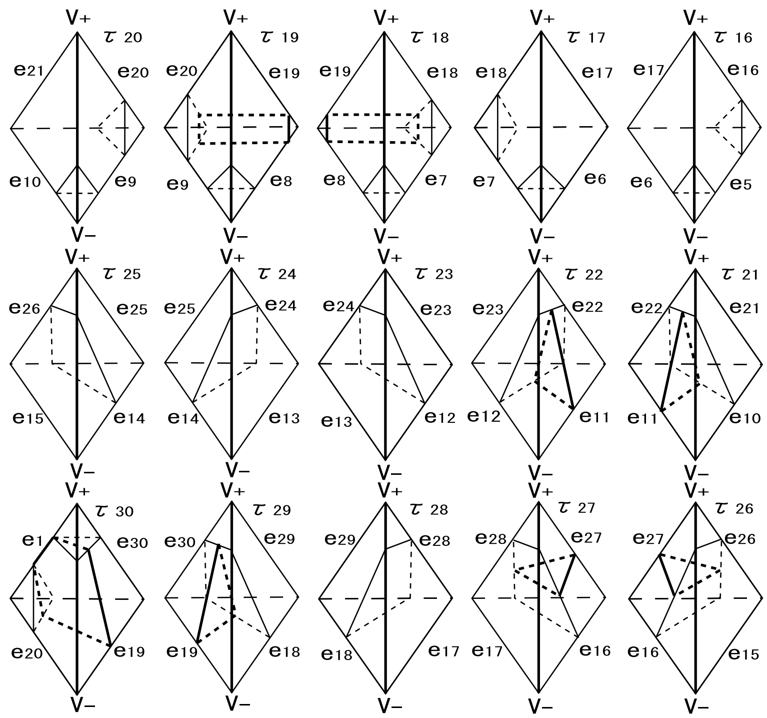

FIGURE 18

0 0 0|0 0 0|0 0 0|0 0 0|0 0 0|0 0 1|0 1 0|0 0 1|0 1 0|0 0 1|0 1 0|0 0 1|0 1 0|0 0 1|0 0 0). This represents a non-orientable surface which is the connected sum of five projective planes. This Q-fundamental surface is, however, not of maximal Euler characteristic. In fact, this surface admits a compressing disk as in Figures 15 and 16. Surgering along this disk, we obtain the Q-fundamental surface

^{t}(0 0 0|0 0 0|1 0 0|1 0 0|0 0 0|0 0 0|0 0 0|0 0 0|0 0 0|0 0 1|0 0 0|0 0 0|0 0 0|1 0 0|1 0 0|
0 0 0|0 0 0|0 0 0|0 0 0|0 0 0|0 0 1|0 0 0|0 0 0|0 1 0|0 0 0|0 0 0|0 0 1|0 1 0|0 0 1|0 0 0)

with four a_k's non-zero. This surface is the connected sum of three projective planes and of maximal Euler characteristic as in [2]. Such a surface is also obtained by another compression along the disk in Figures 17 and 18. The result is the Q-fundamental surface

^{t}(0 0 1|0 0 0|0 0 0|0 0 0|0 0 0|0 0 0|1 0 0|1 0 0|0 0 0|0 0 0|0 0 0|0 1 0|0 0 0|0 0 0|0 0 0|
0 0 0|0 0 0|1 0 0|1 0 0|0 0 0|0 0 0|0 0 0|0 0 1|0 1 0|0 0 1|0 0 0|0 0 0|0 1 0|0 0 0|0 0 0).

The $(418, 153)$-lens space contains a fundamental surface as below which is homeomorphis to the connected sum of five projective planes and has three sheets of normal disks of type X_{k1} for four k's.

${}^t(0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|$
$2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|$
$2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|2, 0, 0|3, 0, 0|3, 0, 0|2, 0, 0|2, 0, 0|2, 0, 0|$
$2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|$
$2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|0, 1, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|$
$1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|$
$1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|2, 0, 0|3, 0, 0|3, 0, 0|$
$2, 0, 0|2, 0, 0|2, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|$
$1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|$
$1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|$
$1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 1|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|$
$0, 0, 0|0, 1, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|$
$1, 0, 0|2, 0, 0|2, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 1|0, 0, 0|0, 0, 0|$
$0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 1, 0|0, 0, 0|0, 0, 0|0, 0, 0|1, 0, 0|1, 0, 0|0, 0, 0|$
$0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 0|0, 0, 1|0, 0, 0|0, 0, 0|0, 1, 0|0, 0, 0|0, 0, 0|0, 0, 1|0, 1, 0|0, 0, 1)$

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