今野　(2011 年 10 月 18 日)

情報統計学の問題 (その１) の解答例

問題 1　記号は講義のものを使用する.

(1) $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}_n^2$ を示せ.

(2) $\sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2 + n(\bar{X}_n - \mu)^2$ を示せ.

(3) $\mathbb{E}[X_i - \bar{X}_n] = 0$ を示せ.

(4) $\mathbb{COV}[\bar{X}_n,\, X_i - \bar{X}_n] = \mathbb{E}[\bar{X}_n(X_i - \bar{X}_n)] = 0$ を示せ.

解答. (1) $\bar{X}_n = \sum_{i=1}^n X_i$ に注意して,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2 &= \frac{1}{n}\sum_{i=1}^n (X_i - 2\bar{X}_n X_i + \bar{X}_n^2) \\ &= \frac{1}{n}\left\{\sum_{i=1}^n X_i - 2\bar{X}_n \sum_{i=1}^n X_i + n\bar{X}_n^2\right\} \\ &= \frac{1}{n}\left\{\sum_{i=1}^n X_i - 2\bar{X}_n(n\bar{X}_n) + n\bar{X}_n^2\right\} = \frac{1}{n}\sum_{i=1}^n X_i - n\bar{X}_n^2\end{aligned}$$

(2)

$$\begin{aligned}\sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n \left\{(X_i - \bar{X}_n) + (\bar{X}_n - \mu)\right\}^2 \\ &= \sum_{i=1}^n \left\{(X_i - \bar{X}_n)^2 + (\bar{X}_n - \mu)^2 + 2(X_i - \bar{X}_n)(\bar{X}_n - \mu)\right\} \\ &= \sum_{i=1}^n (X_i - \bar{X}_n)^2 + \sum_{i=1}^n (\bar{X}_n - \mu)^2 + 2(\bar{X}_n - \mu)\sum_{i=1}^n (X_i - \bar{X}_n)\end{aligned}$$

ここで，$\bar{X}_n = \sum_{i=1}^n X_i$ に注意して，

$$\begin{aligned}2(\bar{X}_n - \mu)\sum_{i=1}^n (X_i - \bar{X}_n) &= 2(\bar{X}_n - \mu)\left\{\sum_{i=1}^n X_i - n\bar{X}_n\right\} \\ &= 2(\bar{X}_n - \mu)\left\{(n\bar{X}_n) - n\bar{X}_n\right\} = 0\end{aligned}$$

となることより等式はわかる.

(3)

$$\mathbb{E}[X_i - \bar{X}_n] = \mathbb{E}[X_i] - \mathbb{E}[\bar{X}_n] = \mu - \mu = 0.$$

(4) 共分散公式

$$\mathbb{COV}[W_1,\, W_2] = \mathbb{E}[W_1 W_2] - \mathbb{E}[W_1]\mathbb{E}[W_2]$$

を利用する：

$$\mathbb{COV}[\bar{X}_n,\, X_i - \bar{X}_n] \;=\; \mathbb{E}[\bar{X}_n(X_i - \bar{X}_n)] - \mathbb{E}[\bar{X}_n]\mathbb{E}[X_i - \bar{X}_n]$$

しかし，(3) より $\mathbb{E}[X_i - \bar{X}_n] = 0$ なので，

$$\mathbb{COV}[\bar{X}_n,\, X_i - \bar{X}_n] = \mathbb{E}[\bar{X}_n(X_i - \bar{X}_n)]$$

つぎに，$X_1,\, X_2,\, \ldots,\, X_n$ は独立なので，$i \neq j$ ならば，

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j]$$

に注意する．

$$\begin{aligned}
\mathbb{E}[\bar{X}_n(X_i - \bar{X}_n)] &= \mathbb{E}[\bar{X}_n X_i - \bar{X}_n^2] \\
&= \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n} X_j X_i\right] - \mathbb{E}[\bar{X}_n^2] \\
&= \mathbb{E}\left[\frac{1}{n}\sum_{j\neq i}^{n} X_j X_i + \frac{1}{n}X_i^2\right] - \left\{\mathbb{E}[\bar{X}_n^2] - (\mathbb{E}[\bar{X}_n])^2 + \mu^2\right\} \\
&= \frac{1}{n}\sum_{j\neq i}^{n} \mathbb{E}[X_j]\mathbb{E}[X_i] + \frac{1}{n}\mathbb{E}[X_i^2] - \left\{\mathbb{E}[\bar{X}_n^2] - (\mathbb{E}[\bar{X}_n])^2 + \mu^2\right\} \\
&= \frac{1}{n}\sum_{j\neq i}^{n} \mu^2 + \frac{1}{n}\left\{\mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 + \mu^2\right\} - \left\{\mathbb{E}[\bar{X}_n^2] - (\mathbb{E}[\bar{X}_n])^2 + \mu^2\right\} \\
&= \frac{1}{n}\sum_{j\neq i}^{n} \mu^2 + \frac{1}{n}\left\{\mathbb{VAR}[X_i] + \mu^2\right\} - \left\{\mathbb{VAR}[\bar{X}_n] + \mu^2\right\} \\
&= \frac{n-1}{n}\mu^2 + \frac{1}{n}\left\{\sigma^2 + \mu^2\right\} - \left\{\frac{\sigma^2}{n} + \mu^2\right\} = 0
\end{aligned}$$